DISTINGUISHING PURE QUANTUM STATES

The project aims at determining the minimal number of measurements that are needed to distinguish any pair of pure quantum states in a finite-dimensional quantum system. This question goes back to Wolfgang Pauli, and has recently been (almost) solved by combining tools from the theory of orthogonal polynomials and differential geometry.

The mathematical formulation of the problem is simple: A d-level quantum system is modelled by the d-dimensional Euclidean space $\mathbb{C}^d$ and quantum states are given by positive semidefinite matrices of trace 1. The set of quantum states is convex and closed and its extreme points are given by the projections onto 1-dimensional subspaces, denoted by $|v\rangle\langle v|$ for some $v \in \mathbb{C}^d$ (acting as $|v\rangle\langle v|w = \langle v, w\rangle v$ for any vector $w \in \mathbb{C}^d$). We call these extreme points pure states. The most basic form of a quantum measurement on such a system is specified by an orthonormal basis $\mathcal{B} = \{w_1, \ldots, w_d\} \subset \mathbb{C}^d$, and for a pure state $|v\rangle\langle v|$ the measurement outcome i is obtained with probability

$$P(i) = |\langle w_i, v\rangle|^2.$$

The question is now how many orthonormal bases $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are needed to determine $|v\rangle\langle v|$ from the statistics of these measurements, i.e., from the probabilities

$$P(i|j) = |\langle w_i^j, v\rangle|^2,$$

where $\mathcal{B}_j = \{w_1^j, \ldots, w_d^j\} \subset \mathbb{C}^d$.

Surprisingly, the answer does not increase when the dimension d is growing. Specifically, $k = 4$ measurements are enough when $d \geqslant 5$ or $d = 3$, and $k = 3$ measurements are enough for $d = 2$. For $d = 4$ the answer is not known, but it is either $k = 3$ or $k = 4$. The objectives of the project are the following:

(1) Learn the basics of quantum mechanics and the formalism of quantum measurements to fully understand the problem.
(2) Present the proof that 4 orthonormal bases are enough to distinguish every pair of pure states.
(3) Understand the reformulation of the problem in terms of smooth embeddings of complex projective spaces.
(4) Optional: Explore the open question of $d = 4$ and/or further problems of a similar flavour.

Prerequesites for the project are a thorough understanding of linear algebra and basic analysis. Knowledge in differential geometry, e.g., know about the projective spaces as smooth manifolds, is useful.

Literature: Claudio Carmeli, Teiko Heinosaari, Jussi Schultz, and Alessandro Toigo, *How many orthonormal bases are needed to distinguish all pure quantum states?*, European Physics Journal D, 69:179, 2015.