The Klein quartic and its n-Weierstrass points

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Project description

The aim of this project is to study a particular algebraic curve, the Klein quartic given by the defining polynomial

$$x^3y + y^3z + z^3x = 0.$$

The Klein quartic has been intensively studied since the 1800s and is well known, but there are still unanswered questions about the curve, see [Fau21; Kle78; Lev99].

In this project the main focus will be on the points where the curve can be approximated unexpectedly well by other curves. If a curve has a point with contact order three with a line at that point, then the point is referred to as an inflection point (1-Weierstrass point) on the curve. The clue is that the tangent line has more contact than expected with the curve at the point. It is well known that the Klein quartic has 24 inflection points and that they have quite special properties. A first goal in this project is to find and describe some of these points and tangent lines.

Similarly, if a curve at a smooth point has contact order six with a conic, the point is referred to as a sextactic point (2-Weierstrass point). The clue is that the so-called osculating conic has more contact than expected with the curve at the point, see [MM19]. The Klein quartic has 84 sextactic points. A second goal in this project is to determine and describe these points and the associated (hyper)osculating conics.

Equivalently, a tentactic point (3-Weierstrass point) is a smooth point where the curve has contact order ten with a cubic curve. On the Klein quartic it is known that the inflection points and the sextactic points are also tentactic points, and it is estimated that the Klein quartic has another 168 3-Weierstrass points that are neither inflection points nor sextactic points [Far10]. Investigating the 3-Weierstrass points of the Klein quartic is the ultimate goal of this project.

This project can go in many different directions after this. One possible task is to get an overview of the n-Weierstrass points for $n \geq 4$. Another task is to study the automorphism group of the Klein quartic and the role of the n-Weierstrass points in this setting. A third task is to study the hyperbolic model of the Klein quartic.

References

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