Projects for MAT2000

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The Atiyah-Singer index theorem [AS68] is about the marriage of topology, geometry and analysis. It is about the relation between the topology of the general linear group $GL(n, \mathbb{C})$, the geometry of differentiable manifolds and the analysis of the behavior of the solutions of elliptic partial differential equations. Such a differential equation carries an "index" – which is an indicator of its being solvable. The content of the Atiyah-Singer index theorem is that the "index" depends on the topology – it is a topological invariant.

It begins with a classical theorem of Riemann and Roch about meromorphic functions, which you must have heard in a complex analysis course, on Riemann surfaces, see [For81] if you are interested. In the 1950's, Friedrich Hirzebruch generalized the Riemann-Roch theorem vastly, making use of landmark geometric theories such as the theory of bordisms and characteristic classes; the book [Hir66] is fantastic but difficult! Around the same time, Alexandre Grothendieck pioneered an entirely new subject called K-theory to prove his version of the Hirzebruch-Riemann-Roch theorem. This is the starting point of the so called integrality theorems in topology. What Michael Atiyah and Isadore Singer discovered is that these integers are none other than, roughly speaking, the number of solutions to certain differential equations!

If you want to read more about the index theorem, you may consult the fantastic paper [Fre21]. There is also the comprehensive book [Pal65] and the excellent book [Gil95]. The paper [AS68] is beautifully written but would be a difficult read!

The projects described below provide the different backgrounds to understand the index theorem. We will follow the beautiful book [BB85] for the three projects; these are the first three parts of this book. The book has curated exercises that will help you a lot.

Project 1: Operators with index

In this project, we will understand what "index" means. This topic falls under the name *functional analysis*. We will start with the basics of functional analysis – Hilbert spaces, Banach spaces, bounded operators and then read Chapters 1–5 and prove that the index is a homotopy invariant. Time permitting, we will also see some of the remaining chapters of Part I of the book. The main goal for you is to be able to deal with bounded operators which are infinite dimensional

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generalizations of matrices that you came across in linear algebra. We will also use the book [RS80] for the basics of functional analysis. If you want to learn more about Fredholm operators, a good book is [Dou98].

Project 2: Analysis on manifolds

In this project, we will understand partial differential operators, following Part II of the book. We will begin with a crash course on manifolds, following Chapter 2 and then read until Chapter 5. Here, the main goal is to get comfortable with basic differential geometry and differential equations on manifolds. If you want to learn more about these things, you may consult the books [Wel80, Pal65].

Project 3: K-theory

The goal of this project is to give you a feel for the vast subject of K-theory and why it is involved in the Atiyah-Singer index theorem. We will first read the Appendix to the book and then come back to Part III, Chapter 1 and finish with a proof of what is called the Bott periodicity theorem – a landmark result. If you want to learn more about K-theory, there is the classic but difficult [Ati67]. I recommend the beautiful book [Kar78].

References

[AS68] M.F. Atiyah and I.M. Singer, *The index of elliptic operators. I*, Ann. of Math. (2) **87** (1968), 484–530. MR 236950

[Ati67] M.F. Atiyah, *K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1967, Lecture notes by D. W. Anderson. MR 224083

[BB85] B. Booss and D.D. Bleecker, *Topology and analysis*, Universitext, Springer-Verlag, New York, 1985, The Atiyah-Singer index formula and gauge-theoretic physics, Translated from the German by Bleecker and A. Mader. MR 771117

[Dou98] R.G. Douglas, *Banach algebra techniques in operator theory*, second ed., Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998. MR 1634900

[For81] O. Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, Springer-Verlag, New York-Berlin, 1981, Translated from the German by Bruce Gilligan. MR 648106

[Fre21] D.S. Freed, *The Atiyah-Singer index theorem*, Bull. Amer. Math. Soc. (N.S.) **58** (2021), no. 4, 517–566. MR 4311554

[Gil95] P.B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, second ed., Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1396308

[Hir66] F. Hirzebruch, *Topological methods in algebraic geometry*, enlarged ed., Die Grundlehren der mathematischen Wissenschaften, vol. Band 131, Springer-Verlag New York, Inc., New York, 1966. MR 202713

[Kar78] M. Karoubi, *K-theory*, Grundlehren der Mathematischen Wissenschaften, vol. Band 226, Springer-Verlag, Berlin-New York, 1978, An introduction. MR 488029

[Pal65] R.S. Palais, *Seminar on the Atiyah-Singer index theorem*, Annals of Mathematics Studies, vol. No. 57, Princeton University Press, Princeton, NJ, 1965, With contributions by M. F. Atiyah, A. Borel, E. E. Floyd, R. T. Seeley, W. Shih and R. Solovay. MR 198494

[RS80] M. Reed and B. Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, Functional analysis. MR 751959

[Wel80] R.O. Wells, Jr., *Differential analysis on complex manifolds*, second ed., Graduate Texts in Mathematics, vol. 65, Springer-Verlag, New York-Berlin, 1980. MR 608414