Combinatorics and analysis behind symmetric groups

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Goal

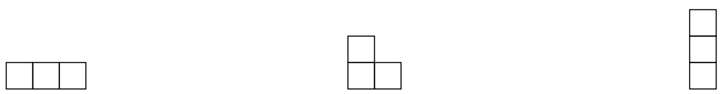
- Low-level: learn manipulation for Young diagrams of large size, and relate them to functions
- High-level: witness an interesting relation between combinatorics, probability theory, linear algebra, and abstract algebra through asymptotic representation theory

Description

The goal of this project is to look at a relation between the symmetric groups S_N (from abstract algebra), random matrices (from probability theory), and the Young diagrams (from combinatorics), and look at the asymptotic behavior as N tends to infinity (methods of analysis).

The starting point is to look at different ways to represent permutations $s \in S_N$ as matrices $X^s = (x^s_{ij})^k_{i,j=1}$ satisfying the multiplicativity condition $X^s X^t = X^{st}$. Then an interesting connection to combinatorics shows up: the fundamental building blocks of such representations are labeled by the Young diagrams of degree N.

A Young diagram of degree N is a shape consisting of N little squares aligned in a 'corner' of a big box, like the following figures. Equivalently, a Young diagram represents a partition $N = a_1 + a_2 + \dots$ with $a_1 \geq a_2 \geq \cdots \geq 0$.



(a) $a_1 = 3$ (b) $a_1 = 2, a_2 = 1$ (c) $a_1 = a_2 = a_3 = 1$

Figure 1: Young diagrams of degree $N = 3$

These objects at different N are related by the natural inclusion $S_N \subset S_{N+1}$ (treat the permutations of $1, \dots N$ as the ones for $1, \dots, N+1$, fixing $N+1$), and inclusion of smaller Young diagrams to bigger ones 'at the corner'. When we look at how Young diagrams grow, we encounter a surprising connection to the theory of random matrices.

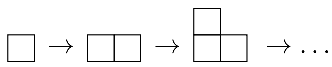


Figure 2: growing Young diagrams (Young tableaux)

A random matrix is a matrix $A = (a_{ij})^n_{i,j=1}$ whose entries a_{ij} are randomly chosen according to some probability distribution. If we impose a condition saying that A is a Hermitian matrix, we can look at the distribution of the eigenvalues, which is again an n-tuple of numbers on the real line following some probability distribution.

Following a survey by P. Śniady [1], we are going to learn how the asymptotics of Young diagrams is related to the eigenvalue distribution of random matrices through the concept of free cumulants. To help the understanding of abstract theory, we try to incorporate numerical experiments using computer algebra (`python` with `numpy` library, or `MATLAB`).

References

[1] Piotr Śniady, *Combinatorics of asymptotic representation theory*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2013, pp. 531–545. (`arXiv:1203.6509`)