MAT2000 PROJECT PROPOSAL
OPTIMAL MATCHING, THEORY AND NUMERICS

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(Complete bipartite) matching problems ask about the best way to pair up two collections of objects of the same cardinality, i.e., in such a way that a cost function is minimised. Matching problems have been given much attention in computer science and statistical physics. For this project, we look at two sets each of n random points taking values in a box $[0,1]^d$ of d dimensions.

Since the original paper of Ajtai–Komlós–Tusnády [1], there have been several proofs (see [5, Section 4.5]) of the matching upper and lower bounds as given by (variations on) the following theorem:

Theorem 1. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independently uniformly distributed on $[0,1]^d$. Define the transportation cost*

$$c_{n,p,d} := \mathbb{E} \min_{\pi \in S^n} \frac{1}{n} \sum_{i=1}^{n} \left| X_i - Y_{\pi(i)} \right|^p ,$$

where S^n is the permutation group on on $\{1, \ldots, n\}$, and the absolute value denotes the d-Euclidean distance. The following asymptotics hold:

$$c_{n,p,d} \sim \begin{cases} n^{-p/2} & d = 1 \\ (\log(n))^{p/2} \, n^{-p/2} & d = 2 \\ n^{-p/d} & d \geq 3 \end{cases} .$$

Here, $a_n \sim b_n$ means that $\limsup_n a_n/b_n$ and $\limsup_n b_n/a_n$ are both bounded.

A particularly short proof was given in [3]. Obviously, the critical behaviour happens in dimension $d = 2$. For the $p = d = 2$ case, it was shown relatively recently in [2] via PDE approach that the limit itself exists (the $\limsup$ matches the $\liminf$ at $(2\pi)^{-1}$). This was based on heuristics given in [4]. Limits are known to exist for $d \geq 3$ when $p < d/2$.

The goal of this project is twofold:

(i) To understand how the proofs of Theorem 1 are related to each other. This will introduce serve to introduce concepts such as empirical measures, Kantorovich duality, among others.

(ii) Numerically to test whether $n^{p/d} c_{n,p,d}$ has a limit as $n \to \infty$ for $d \geq 3$, $p \geq d/2$, and explore analogous questions for $d = 2$, $p \neq 2$. It would be very interesting to be able numerically to see a *rate* of convergence, beyond the limit proven for $d = p = 2$, and to conjecture some lower order terms for $n c_{n,2,2}/\log(n) - (2\pi)^{-1}$ (see also [2, Section 6]).

References

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[2] L. Ambrosio, F. Stra, and D. Trevisan. A PDE approach to a 2-dimensional matching problem. *Probab. Theory Relat. Fields*, 173:433–477, 2019.
[3] S.G. Bobkov and M. Ledoux. A simple Fourier analytic proof of the AKT optimal matching theorem. *Ann. Appl. Probab.*, 31(6):2567–2584, 2021.
[4] S. Caracciolo, C. Lucibello, G. Parisi, and G. Sicuro. Scaling hypothesis for the Euclidean bipartite matching problem. *Phys. Rev. E*, 90:012118, 2014.
[5] M. Talagrand. *Upper and Lower Bounds for Stochastic Processes: Decomposition Theorems.* Volume 60 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics.* Springer, Cham, 2021. Second Edition.

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