PROJECTS FOR MAT2000

TUYEN TRUNG TRUONG

4 projects are offered:

1. Project 1: Solving systems of polynomials, symbolically

Polynomial equations appear quite often in algebra and algebraic geometry. People surely see it at least once in their high school mathematics (quadratic and cubic equations, in one variable), and it is hidden from the side but it has been also used in cryptography (elliptic curves, and here, Niels Henrik Abel played an important role, as he did in the theory of equations of degree 5 or higher). Many questions in mathematics can be reduced to (countably many) polynomial equation systems: from questions in elementary Euclidean geometry which you see in high school (circles, lines, triangles) to long standing open questions such as: the Jacobian conjecture, whether two given algebraic varieties are birational to each other... (One long-standing 1-million-dollar open question, the Hodge conjecture, unfortunately is not yet known to be reducible to countably many polynomial equations. It is very good if you can figure out the answer.)

Jacobian conjecture asks if a polynomial map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, whose Jacobian is every invertible, has a polynomial inverse. It is easy to state but it is very difficult. It is a long standing open question and has relations to many fields in mathematics and physics. Many wrong proofs have been published on it.

Even solving one equation in one complex variable is already challenging and difficult: Think about the Riemann hypothesis.

2 directions can be offered:

Direction 1: Groebner Basis. In this direction, the student will use computer algebra to check whether a polynomial map is an automorphism.

Direction 2: Properness of maps. In this direction, the student will study properness of polynomial maps and applications to Jacobian conjecture.

2. Project 2: Solving systems of polynomials, numerically

The setting is as in Project 1. However, the student will study solving systems of equations using a new variant of Newton's method (which is a combination with Armijo's Backtracking line search) to solve the systems of equations. The advantage here is that one can also solve systems of equations in real variables, and not just complex variables. For example, this way one can try to solve systems of equations where variables are quaternions. Quaternions are quite useful, for example in (surprisingly) electronic toothbrushes!

Date: January 3, 2024.

2010 *Mathematics Subject Classification.* 32-xx,

In this project, students will also study connections to other subjects such as flows (in ODE), Voronoi's diagrams and stochastic processes.

3. Project 3: Optimization in small scale

In this project, the student will study some optimization methods, in small dimensions, which are both rigorously proven, easily implemented and very good experimented. The common theme of these methods are the incorporation of Armijo's Backtracking line search.

4. Project 4: Optimization in large scale and Deep Learning

In this project, the student will study about real use of optimization in large scale, such as in training Deep Neural Networks (Deep Learning). Deep Learning appears more and more in many aspects of life, and you can already see its usage on your smart phones.

References: (More will be given later)

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