MAT2000 PROJECTS

Valerio Proietti

valeriop@math.uio.no https://vproietti.gitlab.io/

Project proposals for MAT2000 (Spring 2024).

THE INVARIANT SUBSPACE PROBLEM

This problem belongs to the field of mathematics known as *functional analysis*, and more specifically *operator theory*. It is a partially unresolved problem asking whether every bounded operator on a complex Banach space sends some non-trivial closed subspace to itself. Many variants of the problem have been solved, by restricting the class of bounded operators considered or by specifying a particular class of Banach spaces. The problem is still open for separable Hilbert spaces, although in May 2023 a preprint by Enflo [8] appeared on `arXiv` which, if correct, solves the problem for Hilbert spaces.

This problem asks a very natural question (a basic property in the structure theory of linear operators), it is extremely simple to state, but it appears to be quite hard to solve. This feature is shared by some of the most well-known problems in mathematics.

Project plan. We start with preliminaries such as dual spaces, weak topologies, and Banach–Alaoglu theorem. We continue with studying the proofs of two positive results: the Aronszajn-Smith Theorem on invariant subspaces of completely continuous operators [1], and the Lomonosov Theorem [11], one of the most famous results on this problem, about the existence of invariant subspaces for operators in the centralizer of a compact operator.

We then tackle the spectral theorem and existence of invariant subspaces for operators which commute with normal operators. Extra material could be the counterexample to the problem in ℓ^1 by Read [14], and the recent preprint by Enflo.

Other references. [2, 4, 7, 10, 12, 15, 17, 18].

THE OKA-CARTAN THEOREMS

The Oka–Cartan fundamental theorem on Stein manifolds (also known as Cartan's *Theorem B*) concerns the cohomology of coherent sheaves (i.e., a generalization of vector bundles) on a Stein manifold (i.e., a generalization of the notion of "holomorphically convex domain" in $\mathbb{C}^n$). This result is significant both as applied to several complex variables, and in the general development of sheaf cohomology.

This theorem implies several other interesting and important results. First, it implies the so-called Cartan's *Theorem A* [3] on "global–point generation" for coherent sheaves on Stein manifolds. It also provides solutions to the first and second Cousin problems on Stein manifolds (concerning the existence of meromorphic functions that are specified in terms of local data) [5]. Another corollary is the *analytic* de Rham Theorem, namely the fact that standard de Rham cohomology can be computed via analytic forms on a Stein manifold. More generally, the Oka–Cartan fundamental theorem plays a key role in the GAGA correspondence of Serre [16], which enables the transfer of results from complex analytic geometry to algebraic geometry over $\mathbb{C}$.

Project plan. We start with preliminaries such as the notion of coherent sheaf, sheaf resolutions, Runge's and Hartogs' Theorems, and holomorphic convexity.

We continue with the proof of the Oka–Cartan fundamental theorem on manifolds of increasing complexity: convex cylinder domains, analytic polyhedra, holomorphically convex domains, and finally Stein manifolds. Finally we focus on the consequences of this result as described above.

Other references. [6, 9, 13].

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