Understanding Wave-Breaking Phenomena in the Camassa-Holm Type Equation – A User-Friendly Approach

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1 Background Review 1: Blow-up of Riccati-type Equations

Let $a > 0$. Consider the Riccati-type Ordinary Differential Equations (ODEs):

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = aX^2(t), \quad X(0) = X_0 \in \mathbb{R}, \tag{1.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = -aY^2(t), \quad Y(0) = Y_0 \in \mathbb{R}. \tag{1.2}$$

Blow-up refers to the solution tending to infinity in finite time. The discussion in this part will delve into how X_0 and Y_0 lead to the blow-up of solutions X and Y, respectively.

2 Background Review 2: Introduction of Camassa-Holm Equation to Students

The Camassa-Holm (CH) equation is defined as

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{2.1}$$

This equation, derived by [1], serves as a model governing shallow water waves, with the unknown $u = u(t,x)$ representing the velocity. I will provide a concise introduction to the following aspects for undergraduate students:

- Key features of the CH equation[1];
- The concept of wave-breaking and its significance in the context of nonlinear partial differential equations.

3 Target: Wave-breaking in the CH Type Equation

In this brief introduction, for the CH equation, I outline that one can find some quantity $M(t)$, as per [2], such that

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) \leq \beta N - \beta\frac{1}{2}M^2(t). \tag{3.1}$$

This constitutes a Riccati-type inequality. Drawing from the analysis of blow-up in Riccati-type equations, students are encouraged to employ similar techniques to construct blow-up solutions in the CH equation. The primary **tasks** for students include:

- Understanding the application of blow-up techniques (based on (1.1), (1.2)) in (3.1);
- Learning the derivation of the breaking rate from (3.1);
- Acquiring the skills to derive (3.1) in analogous models.

By accomplishing these tasks, students are anticipated to become familiar with advanced results in current literature and gain the ability to enter the corresponding field.

4 Summary and necessary background

The project is tailored to be accessible and engaging for individuals interested in analysis and differential equations. It is strongly suggested that the prospective student are familiar with the existence theory of ODEs in $\mathbb{R}^d$ and basic knowledge of Hilbert spaces. Upon completion, students are expected to be more acquainted with advanced mathematical concepts and techniques, including wave-breaking, blow-up analysis, convolution, Sobolev spaces, among others.

It is highly recommend that potential students should have taken MAT2400 and/or MAT3400, and have taken (or take in parallel) MAT3360 and/or MAT4301.

[1] As the emphasis of this study is on blow-up phenomena, certain useful results pertaining to the CH equation will be accepted without providing a proof.

References

[1] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.

[2] A. Constantin. On the blow-up of solutions of a periodic shallow water equation. *J. Nonlinear Sci.*, 10(3):391–399, 2000.