Bridging Dimensions: A Study on the Existence Theory of ODEs/PDEs

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1 Background Review: ODE in $\mathbb{R}^d$

Consider the Ordinary Differential Equation (ODE)

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = b(t, X(t)), \quad X(0) = X_0 \in \mathbb{R}^d, \tag{1.1}$$

where b is the coefficient $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$.

The classical existence theory for (1.1) asserts that a locally Lipschitz condition on $b(t, \cdot)$ implies the existence and uniqueness of a local-in-time solution. Additional conditions such as linear growth can ensure the existence of such a solution for all time, i.e., a globally existing solution.

2 Target 1: ODE in Hilbert Space $\mathbb{H}$

To extend the previous theory to infinite-dimensional spaces, we consider

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = b(t, X(t)), \quad X(0) = X_0 \in \mathbb{H}, \tag{2.1}$$

where $\mathbb{H}$ is a separable Hilbert space, encompassing the finite-dimensional space case $\mathbb{R}^d$. Similar results will be derived in this case, with emphasis on the distinctions that arise.

3 Target 2: Application to Certain PDEs

With the theory developed in **Target 1**, we can handle certain partial differential equations (PDEs). For instance, consider the following SPDE:

$$\partial_t u = (\mathbf{I} - \partial_x^2)^{-1}(u^2 + u_x^2), \quad u(0) = u_0 \in H^s, \tag{3.1}$$

where H^s is the Sobolev space with index s and $\mathbf{I}$ is the identity mapping. We will explain how this can be covered by the theory in **Target 1** and how to verify the conditions outlined in **Target 1**.

4 Possible Target 3: Solving Fluid-type Equations

Finally, contingent upon the student's progress and feedback, there exists the potential to delve deeper in this direction. The groundwork laid in **Target 1** serves as the initial stride towards tackling real-world physical partial differential equations (PDEs), exemplified by the Burgers' equation

$$u_t = -uu_x, \quad u(0) = u_0 \in H^s. \tag{4.1}$$

However, direct application of **Target 1** to solve (4.1) is unfeasible due to the non-invariance of uu_x in H^s, rendering (4.1) incapable of being regarded as an ordinary differential equation (ODE) in H^s. In this phase, we will elucidate how to amalgamate the principles from **Target 1** with the *mollifying method* to effectively address (4.1).

5 Summary and necessary background

This project is designed to be an engaging and accessible journey for those intrigued by analysis and differential equations. It serves as a significant conduit, transitioning learners from the realms of calculus and elementary analysis to the more complex territories of functional analysis and the study of infinite-dimensional spaces. Upon completion, students are expected to be more familiar with some advanced mathematical concepts/techniques, including but not limited to, ODEs in Hilbert spaces, mollifier and Sobolev spaces.

It is highly recommend that potential students should have a solid understanding of the existence theory of ODEs and a foundational knowledge of Hilbert spaces. More specific, the student should have taken MAT2400 and/or MAT3400. It is better that the students have taken MAT3360 and/or MAT4301.

These objectives are interrelated and aligned as follows:

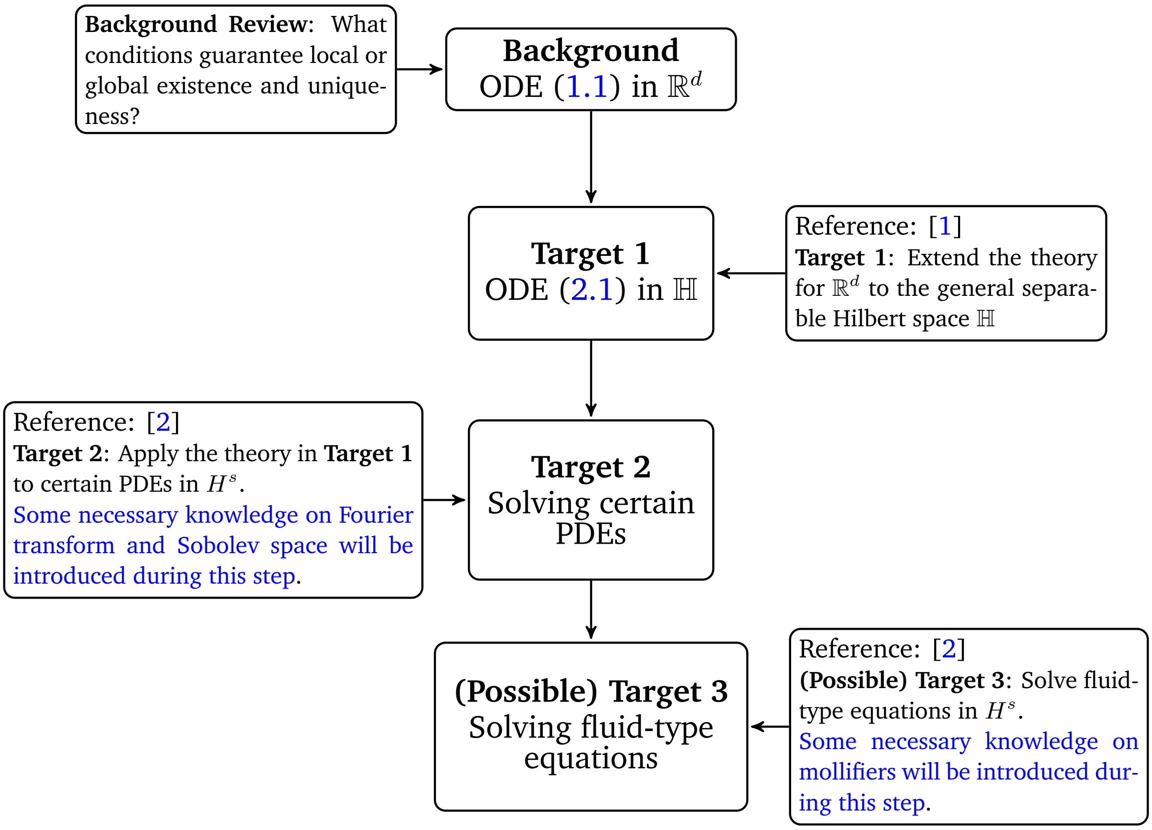

Figure 1: A brief summary of Target 1 and Target 2.

References

[1] J. Dieudonné. Foundations of modern analysis. pages xviii+387, 1969. Enlarged and corrected printing, Pure and Applied Mathematics, Vol. 10-I.

[2] A. J. Majda and A. L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.