1.8.4 b) Show if $f: A \to B$ has an inverse function, then it is 1-1 and onto

(1-1) Assume $f(a) = f(a')$, then $f^{-1}(f(a)) = f^{-1}(f(a'))$

so $a = a'$

(onto) Let $b \in B$, and define $a := f^{-1}(b)$.

Then $f(a) = f(f^{-1}(b)) = b$, so b is in the image of f.

Since b was arbitrary, f is onto

c) Show if $f: A \to B$ is 1-1 and onto then $f^{-1}: B \to A$ is also 1-1 and onto

f is the inverse function of f^{-1}, so

we can apply b) to see that f^{-1} is 1-1 and onto

1.8.5 State which of the following functions are 1-1 and onto

a) $f: \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^6$

Not onto: no x in $\mathbb{Z}$ such that $x^6 = -1$

Not 1-1 $f(1) = 1^6 = 1 = (-1)^6 = f(-1)$

b) $f: \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 3x$

Not onto: No $x \in \mathbb{Z}$ s.t. $3x = 1$

1-1: Since if $f(x) = f(x')$, then $3x = 3x'$

so $x = x'$ (cancellation)

c) $f: \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x - 3$

onto and 1-1. We can check that $f^{-1}(x) = x+3$

is the inverse function of $f(x)$

$f^{-1}(f(x)) = (x-3)+3 = x$

$f(f^{-1}(x)) = (x+3)-3 = x$ Then apply 1.8.4 b)

1.8.10 Show that the inverse image has the following properties for $f: S \to T$ and $A, B \subset T$

a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Check $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

Let $s \in f^{-1}(A \cup B)$, then $f(s) \in A \cup B$,

$\Rightarrow f(s) \in A$ or $f(s) \in B$

$\Rightarrow s \in f^{-1}(A)$ or $f \in f^{-1}(B)$

$\Rightarrow s \in f^{-1}(A) \cup f^{-1}(B)$

Check $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

Let $s \in f^{-1}(A) \cup f^{-1}(B)$, then

$f(s) \in A$ or $f(s) \in B \Rightarrow f(s) \in A \cup B$

$\Rightarrow s \in f^{-1}(A \cup B)$

Since $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$ and

$f^{-1}(A \cup B) \supseteq f^{-1}(A) \cup f^{-1}(B)$ we must have $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

① $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$

Let $s \in f^{-1}(A \cap B)$. Then $f(s) \in A \cap B$

$\Rightarrow f(s) \in A$ and $f(s) \in B$

$\Rightarrow s \in f^{-1}(A)$ and $s \in f^{-1}(B) \Rightarrow s \in f^{-1}(A) \cap f^{-1}(B)$

② $f^{-1}(A \cap B) \supseteq f^{-1}(A) \cap f^{-1}(B)$

Let $s \in f^{-1}(A) \cap f^{-1}(B) \Rightarrow f(s) \in A$ and $f(s) \in B$

$\Rightarrow f(s) \in A \cap B \Rightarrow s \in f^{-1}(A \cap B)$

By ① and ② we must have $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

1.8.12 Use mathematical induction to prove

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Base case: $n=0$

$$(x+y)^0 = 1 = \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k}$$

[Convention: $\binom{0}{0} = 1$]

Assume true for n, check for $n+1$

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= x \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}$$

$$= \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}$$

Convention: $\binom{n}{-1} = \binom{n}{n+1} = 0$

$$= \sum_{n=0}^{n+1} \left[\binom{n}{k-1} + \binom{n}{k}\right] x^k y^{n+1-k} = \sum_{n=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

2.1.3 Finish the multiplication table for D_3

	I	R	R^2	F	RF	R^2F
I	I	R	R^2	F	RF	R^2F
R	R	R^2	I	RF	R^2F	F
R^2	R^2	I	R	R^2F	F	RF
F	F	R^2F	RF	I	R^2	R
RF	RF	F	R^2F	R	I	R^2
R^2F	R^2F	RF	F	R^2	R	I

Examples $\underbrace{(RF)R = R(FR) = R(R^2F) = R^3F = F}$

$$RFR^2 = (RFR)R = FR = R^2F$$

$$F(RF) = (FR)F = (R^2F)F = R^2$$

2.1.5 Is the following table the multiplication table for a group of order 4

*	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	d	a	d

No. Let a^{-1} be the inverse of a.

Since $a * a = a$, $\quad a * a * a^{-1} = a * a^{-1}$

$\Rightarrow a = id$

Similarily, since $b * a = a$ $\quad b * a * a^{-1} = a * a^{-1}$

$\Rightarrow \quad b = id$

$c * a = a \Rightarrow c = id$

$d * a = a \Rightarrow d = id$

So if this is a group $a = b = c = d = id$, so it must have order 1

2.1.6 State which of the following are groups and why

a) $\mathbb{Z}$ under addition

Yes compare axioms for addition with goup axioms (p. 10 and p. 43)

b) $\mathbb{Z}$ under multiplication

No. Since $1 \cdot 0 = 2 \cdot 0 = 0$, 0 cannot have an inverse (Then $1 \cdot 0 \cdot 0^{-1} = 2 \cdot 0 \cdot 0^{-1} \Rightarrow 1 = 2$ contradiction)

c) $\mathbb{R}$ under addition

Yes. Compare axioms

d) $\mathbb{R}$ under multiplication

No. same reason as b)

e) $\mathbb{R} \setminus \{0\}$ under multiplication

Yes. Compare axioms

f) $\mathbb{Z} \setminus \{0\}$ under multiplication

No, since $2 \cdot x = 1$ has no solution in $\mathbb{Z} \setminus \{0\}$, inverses do not exist.

2.1.11. Define the operation $\star$ on $\mathbb{Z}$ by

$a \star b = a - b$. Does this make $\mathbb{Z}$ into a group?

No, since the operation $\star$ is not associative

$$(a \star b) \star c = (a-b)-c$$

$$a \star (b \star c) = a-(b-c) = (a-b)+c$$

$\star$ has an "identity from the right" since

$$a \star 0 = a - 0 = a$$

but <u>not</u> an "identity from the left", since

$$0 \star a = -a \neq a.$$

So in fact $\mathbb{Z}$ with the operation $\star$ also has no identity element.