3.3.11 Show that given a group G and subgroups H, K, we can define an equivalence relation on G by saying $y \sim x$ if $\exists h \in H, k \in K$ such that $y = hxk$. What are the equivalence classes

Reflexive: $x = id\, x\, id = hxk$, since $id \in H, id \in K$

Symmetric: if $y \sim x$, then $y = hxk$, so

$x = h^{-1} y k^{-1}$, so $x \sim y$

since $h^{-1} \in H, k^{-1} \in K$

Transitive: Assume $x \sim y$, $y \sim z$

so $x = hyk$, $y = h' z k'$

then $x = \underbrace{hh'}_{\in H} z \underbrace{k'k}_{\in K}$ so $x \sim z$

The equivalence classes are the double cosets $H \backslash G / K$

3.6.3 Assume $\gcd(m,n)=1$. Define $f: \mathbb{Z} \to \mathbb{Z}_m \oplus \mathbb{Z}_n$

by $f(x) = (x + m\mathbb{Z},\ x + n\mathbb{Z})$

and show that $f(x+y) = f(x) + f(y)$. Then show that $\mathbb{Z}_m \oplus \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$

$f(x+y) = (x+y+m\mathbb{Z},\ x+y+n\mathbb{Z}) = (x+m\mathbb{Z},\ x+n\mathbb{Z}) + (y+m\mathbb{Z},\ y+n\mathbb{Z})$

$= f(x) + f(y)$

So f is a surjective group homomorphism.

We compute $\ker f$

$x \in \ker f$ iff $m \mid x$ and $n \mid x$

$\Leftrightarrow \operatorname{lcm}(m,n) \mid x$

$\Leftrightarrow mn \mid x$ since $\gcd(m,n) = 1$

$\Rightarrow \ker f = mn\mathbb{Z}$

By the 1st isomorphism theorem

$\operatorname{im}(f) \cong \mathbb{Z}/\ker f$ i.e. $\mathbb{Z}_m \oplus \mathbb{Z}_n \simeq \mathbb{Z}/mn\mathbb{Z} = \mathbb{Z}_{mn}$

3.6.5 4 pairwise non-isomorphic groups of order 8

Group	Distinguishing property
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	All non-identity elements have order 2, Abelian
$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	Abelian, contains elements of order 4, but no element of order 8
$\mathbb{Z}_8$	Abelian, contains an element of order 8
D_4	Non-Abelian

The final group of order 8 is the quaternion group

3.66 Are the following statements true or false?

a) If G and H are groups, then $G \oplus H \simeq H \oplus G$

True: $(g,h) \mapsto (h,g)$ is an isomorphism

b) The order of $(3,2)$ in $\mathbb{Z}_{12} \oplus \mathbb{Z}_4$ is 4

True $4(3,2) = (12,8) = (0,0)$

no lower multiple is zero

c) $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ has exactly six subgroups of order 5

True Any subgroup is cyclic generated by say (a,b)

(a,b) and (a',b') generate the same subgroup

$(a',b') \in \langle (a,b) \rangle$ and $(a',b') \neq 0$

so each subgroup has 4 generators.

So the statement is true since $\underbrace{6}_{\text{number of subgroups}} \cdot \underbrace{4}_{\text{generators in each subgroup}} + \underbrace{1}_{\text{the identity}} = \underbrace{25}_{= |\mathbb{Z}_5 \oplus \mathbb{Z}_5|}$

3.6.6 The group $\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $a, b \in \mathbb{Z}$ under matrix multiplication is isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_n$ for any $n \geq 2$

True Since $\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+c & b+d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The function $\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto (a, b)$

is a group isomorphism

3.6.7. Is $|ab| = |a||b|$ for a, b in any finite group G

No. Pick $b = a^{-1}$ for any non-identity a in any G

3.6.15 a)

Suppose $a, b \in G$. G abelian and $|a|, |b| < \infty$ with $\gcd(|a|, |b|) = 1$.

Show that $|ab| = |a||b|$

Step ① Let $|a| = k$, $|b| = \ell$

If $n = xk = y\ell$ is a common multiple of $|a|$ and $|b|$

then $(ab)^n = a^n b^n = e^x e^y = e$.

Therefore $(ab)^{\mathrm{lcm}(k,\ell)} = e$, so $|ab| \mid \mathrm{lcm}(k, \ell)$

since $|a|$ and $|b|$ are coprime we get $|ab| \mid |a||b|$

Step ② Let n be the order of ab

Then $e = e^k = (ab)^{nk} = a^{nk} b^{nk} = b^{nk}$ since $a^k = e$

So the order of b divides nk i.e.

$\ell \mid nk$. Since ℓ and k are coprime $\ell \mid k$

Similarly $e = e^{l} = (ab)^{nl} = a^{nl} b^{nl} = a^{nl}$

So $k \mid nl$ which implies $k \mid n$ since k, l are coprime.

So $k \mid n$ and $l \mid n$

$\Rightarrow kl \mid n$ since $k\ l$ are coprime

So since $|ab| \,\big|\, |a||b|$ and $|a||b| \,\big|\, |ab|$,

we must have $|ab| = |a||b|$

3.6.15 b)

Is it true under these hypotheses that

$\langle a, b\rangle \cong \langle a\rangle \times \langle b\rangle$?

Yes. We prove that

$$\langle a\rangle \oplus \langle b\rangle \longmapsto \langle a, b\rangle$$

$$a^i, b^j \mapsto a^i b^j$$

is an isomorphism. It is clearly surjective, so we must prove that it is injective

Assume $a^i b^j = 0$ (in G) We can assume

$0 \leq i < |a|$, $0 \leq j < |b|$

If i or $j = 0$, then both are 0, so assume $1 \leq i, j$

So $a\, a^{i-1} b^j = 0 \Rightarrow a^{-1} = a^{i-1} b^j$

Since $|a^{-1}| = |a|$, $|a| = |a^{i-1} b^j| = |a^{i-1}|\,|b^j|$

Since $\gcd(|a|, |b|) = 1$, The only possibility is that $|b^j| = 1$, i.e. $b^j = id$

Then we must also have $a^i = id$

3.7.1 Show that G acts on $\{f : G \to \mathbb{C}\}$ by $(L_g f) = f(g^{-1}x)$, Find the analogous right action

$$L_g(L_h f) = f(h^{-1}g^{-1}x) = L_{gh} f$$

$$L_e f = f(x)$$

The right action is

$$f R_g = f(gx)$$

since $f R_g R_h = f(ghx) = f R_{gh}$

3.7.9 Find the orbit of

$$P(x_1,x_2,x_3,x_4) = x_1x_3 - x_2x_4$$

under S_4. What is the order of $Stab(P)$

$$Orb(P) = \{P(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) \mid \sigma \in S_4\}$$

$$= \{x_{\sigma(1)}x_{\sigma(3)} - x_{\sigma(2)}x_{\sigma(4)} \mid \sigma \in S_4\}$$

$$= \{x_1x_3 - x_2x_4,\ x_1x_4 - x_2x_3,$$

$$x_2x_4 - x_1x_3,\ x_2x_3 - x_1x_4,$$

$$x_3x_4 - x_1x_2,\ x_1x_2 - x_3x_4\}$$ 6 elements

Since $|Orb(P)| = |G| / |Stab(P)|$

$|Stab(P)| = 4$

Specifically $Stab(P) = \{(13), (24), (13)(24), id\}$

3.7.11. Find the order of the group of motions of the tetrahedron by computing $|Orb(f)|$ and $|Stab(f)|$ for any face f.

$|Orb(f)| = 4$, since any face can be moved to any other

$|Stab(f)| = |S_3| = 6$ since any symmetry of f is achievable by a motion (including reflections)

So $|G| = |Orb(f)||Stab(f)|$

$= 24$