3.6.13 Show that $G \oplus H$ is commutative iff G and H are commutative

If $G \oplus H$ is commutative, then

$$(g,h)+(g',h')=(g+g',h+h')=(g',h')+(g,h)$$

Hence $g+g'=g'+g$ and $h+h'=h'+h$ for all $g,g' \in G$, $h,h' \in H$, so G and H are commutative

If G, H are commutative, then

$$(g,h)+(g',h')=(g+g',h+h')=(g'+g,h'+h)$$
$$=(g',h')+(g,h)$$

so $G \oplus H$ is commutative

3.6.16 If $g \in G$ and $|g| = r < \infty$,

Show that $|g^t| = \frac{r}{\gcd(r,t)} = \frac{\operatorname{lcm}(r,t)}{t}$

$(g^t)^i = g^{it}$

so $(g^t)^i = e \iff r \mid it$

Therefore, the smallest i satisfying $(g^t)^i = e$

is such that $\operatorname{lcm}(r,t) = it$, i.e. $i = \frac{\operatorname{lcm}(r,t)}{t}$

3.6.17 Suppose $a, b \in G$. G abelian

$|a| = k$, $|b| = \ell$. Show there exists an element $c \in G$

of order $\mathrm{lcm}(k, \ell)$

Define $a^r b^s$ as in the hint.

By 3.5.16

$$|a^r| = \frac{k}{\gcd(k,r)} = \frac{k}{\prod_{i \in I} p^{e_i}}$$

$$|b^s| = \frac{\ell}{\gcd(\ell,s)} = \frac{\ell}{\prod_{i \in J} p_i^{f_i}}$$

By construction these orders are coprime so by 3.5.15 a)

$$|a^r b^s| = \frac{k\ell}{\prod_{i \in I} p^{e_i} \prod_{i \in J} p_i^{f_i}} = \frac{k\ell}{\gcd(k,\ell)} = \mathrm{lcm}(k,\ell)$$

3.7.25 Show that if G is a group with center Z, then Z is a normal subgroup

$$Z = Z(G) = \{ z \in G \mid gz = zg \ \forall g \in G \}$$

So

$$gZg^{-1} = \{ gzg^{-1} \mid gz = zg \ g \in G \}$$

$$= \{ zgg^{-1} \mid gz = zg \ g \in G \}$$

$$= \{ z \mid gz = zg \ g \in G \} = Z$$

Hence Z is normal

3.7.26 Show that if $G/Z(G)$ is cyclic, then G is abelian

Let $g, g' \in G$, and let aZ be a generator of $G/Z(G)$

Write $g = a^i z$ $\quad g' = a^j z'$

Then

$$gg' = a^i z a^j z' = a^i a^j z z' = a^{i+j} z z'$$

$$g'g = a^j z' a^i z = a^j a^i z' z = a^{i+j} z z'$$

So G is abelian

4.5.5 Show there are only two groups of order 10

$\mathbb{Z}_{10}$ and D_5 are two groups of order 10

Let G be a group of order 10

G contains elements of order 5 and 2 (Cauchy's theorem)

If G is abelian, G is cyclic (by previous exercise)

If R is an element of order 5, $\langle R \rangle$ is normal and

$G/\langle R \rangle$ is cyclic of order 2, generated by $F\langle R \rangle$

So $G = \{e, R, R^2, R^3, R^4, F, FR, FR^2, FR^3, FR^4\}$

$F^2 = e$, since $F^2 \in \langle R \rangle$ and F cannot have order 10, since then the group would be cyclic

By Sylow's 3rd theorem, the number of subgroups of order 2 is either 1 or 5.

Case 5 Sylow 2-groups.

F, FR, FR^2, FR^3, FR^4 must all have order 2

If $(FR)^2 = I$ $FRFR = I \Rightarrow RFR = F \Rightarrow FR = R^4F$ So $G = D_5$

The only remaining possibility to exclude is that

$\{I, FR, FR^2, FR^3, FR^4\}$ is a cyclic subgroup.

This is impossible since $F\langle R\rangle$ has order 2 in

$G/\langle R\rangle$, hence $(FR)^2 \in \langle R\rangle$

4.5.11 Show that D_3 is isomorphic to $Aff(3)$

$$Aff(3) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{Z}_3 \right\}$$ with the group operation of matrix multiplication

Map $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mapsto F$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto R$

Check that $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$F \cdot R \quad = \quad R^2 \; F$$