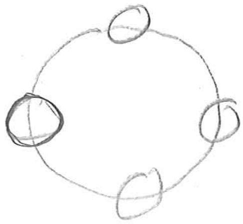
3.7.18 Find the number of bracelets of four beads with two colors $\{A,B\}$



Burnsides Lemma:

$$\#\text{orbits} = \frac{1}{|G|} \sum_{\sigma \in G} \text{Fix}(\sigma)$$

$G = \mathbb{Z}_4 = \{0,1,2,3\}$

G acts on the set $\{AAAA, AAAB, AABA, AABB, \dots, BBBB\}$ of size 2^4

$|\text{Fix}\, 0| = 2^4$ $\quad |\text{Fix}\, 1| = 2$ $\quad |\text{Fix}\, 2| = 2^2$ $\quad |\text{Fix}\, 3| = 2$

all bracelets

single color

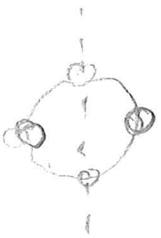
$$\#\text{orbits} = \frac{1}{4}\left(2^4 + 2 + 2^2 + 2\right) = \frac{24}{4} = 6$$

3.7.18. If instead we use the group D_4

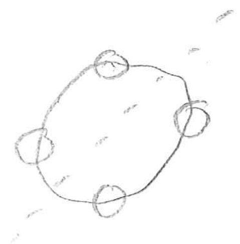
The computations become $\quad D_4 = \{I, R, R^2, R^3, F, FR, FR^2, FR^3\}$

σ	$\lvert \mathrm{Fix}\,\sigma \rvert$
e	2^4
R	2
R^2	2^2
R^3	2
F	2^3
FR	2^2
FR^2	2^3
FR^3	2^2

Can freely choose the color of 2 or 3 beads depending on which flip



3 choices



2 choices

$$\#\text{ orbits} = \frac{1}{8}\left(2^4+2+2^2+2+2^3+2^2+2^3+2^2\right)$$

$$= 6$$

Coincidentally this is the same number of different bracelets

3.7.19 How many ways to paint a square floor with nine tiles using purple and orange paint

P	P	O
O	P	P
P	O	P

Group $G = D_4$

$\{I, R, R^2, R^3, F, FR, FR^2, FR^3\}$

$|\text{Fix } I| = 2^9$

$|\text{Fix } F| = 2^6$

O	P	O
O	P	O
O	P	O

$\shortparallel$

$|\text{Fix } FR^2|$

$|\text{Fix } R| = 2^3$

O	P	O
P	O	P
O	P	O

$\shortparallel$

$|\text{Fix } R^3|$

$|\text{Fix } FR| = 2^6$

O	O	P
O	P	O
P	O	O

$\shortparallel$

$|\text{Fix } FR^3|$

$|\text{Fix } R^2| = 2^5$

$$\# \text{orbits} = \frac{1}{8}\left(2^9 + 2 \cdot 2^3 + 2^5 + 2 \cdot 2^6 + 2 \cdot 2^6\right)$$

$$= \frac{1}{8}(816) = 102$$

3.7.20 In how many ways can you color a cubes faces with 4 colors

Symmetry group of a cube is

S_4 (acting on the diagonals)

Symmetry	permutation	number	$\lvert \mathrm{Fix}\,\sigma \rvert$
identity	e	1	4^6
edge-midpoint rotation	$(12), (13), \dots, (34)$	6	4^3
face-midpoint rotation	$(1234), (1432), \dots, (1342)$	6	4^3
square of face-midpoint rotation	$(12)(34), (13)(24), (14)(23)$	3	4^4
diagonal rotation	$(123), \dots, (243)$	8	4^2

$$\# \text{orbits} = \frac{1}{24}\left(4^6 + 6 \cdot 4^3 + 6 \cdot 4^3 + 3 \cdot 4^4 + 8 \cdot 4^2\right) = 240$$

3.7.24 Find all conjugacy classes in D_3, then check the class equation

$$D_3 = \{I, R, R^2, F, FR, FR^2\}$$
$$\cong$$
$$S_3 = \{e, (123), (132), (12), (13), (23)\}$$

$\{e\}$ is a conjugacy class since $geg^{-1} = e \quad \forall g \in D_3$

$\{(123), (132)\}$ is a conjugacy class, since

$$(23)(123)(23) = (132)$$

and conjugation preserves cycle length, so no length 2 cycle is in the class

$\{(12), (13), (23)\}$ is a conjugacy class since

$$(123)(12)(132) = (1)(23) = (23)$$

$$(123)(23)(132) = (13)$$

Class equation $\quad 6 = 1 + 2 + 3$

4.2.4 Let a be an element of a finite abelian group G under addition. Define the function

$$\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases} \qquad \text{Let } n = |G|$$

a) Show $\delta_a * \delta_b = \frac{1}{n} \delta_{a+b}$

b) Suppose $f : G \to \mathbb{C}$ Show that $(f * \delta_a)(x) = \frac{1}{n} f(x-a)$

a) $$\delta_a * \delta_b(x) = \frac{1}{n} \sum_{g \in G} \underbrace{\delta_a(x-g)\delta_b(g)}_{= \begin{cases} 1 & \text{if } g = b \;\; x-g = a \\ 0 & \text{otherwise} \end{cases}}$$

$$= \frac{1}{n} \delta_{a+b}$$

b) $$(f * \delta_a)(x) = \frac{1}{n} \sum_{g \in G} f(x-g)\,\delta_a(g)$$

$$= \frac{1}{n} f(x-a)\,\delta_a(a) = \frac{1}{n} f(x-a)$$

4.2.5 a) Show that $L^2(\mathbb{Z}_n)$ can be identified with $\mathbb{C}^n$

In fact show that $\{\delta_a\}_{a\in\mathbb{Z}_n}$ form a basis.

The functions δ_a are linearly independent:

Assume $\sum_{a\in\mathbb{Z}_n} c_a \delta_a = 0$

Then $\left(\sum_{a\in\mathbb{Z}_n} c_a \delta_a\right)(g) = 0 \Rightarrow c_g = 0$ since $\left(\sum_{a\in\mathbb{Z}_n} c_a\delta_a\right)(g) = c_g$

this applies for all $g\in G$, so all coefficients $c_g = 0$ $\forall g\in G$

The functions δ_a span $L^2(G)$

Let $f: \mathbb{Z}_n \to \mathbb{C}$ be a function

then $f = \sum_{g\in\mathbb{Z}_n} f(g)\delta_g$

b) The space $L^2(\mathbb{Z}_n)$ has an identity element e under convolution

Pick $e = n\delta_0$

$$(n\delta_0 * f)(x) = \frac{1}{n} \sum_{g\in\mathbb{Z}_n} n\delta_0(x-g) f(g) = f(x)$$

c) Is $L^2(\mathbb{Z}_n)$ a group under convolution

No in 4.2.3 we check that $*$ is commutative, distributive and associative

Remains to check existence of inverses, which will turn out to be false

Consider the element $\sum_{a\in G} \delta_a$

Assume $\sum_{a\in G} \delta_a * \sum_{g\in G} x_g \delta_g = e = n\delta_0$

Then $\sum_{g\in G} x_g \sum_{a\in G} \delta_{a+g} = n\,\delta_0$

But the left side is the constant function $\sum_{g\in G} x_g$, a contradiction

5.2.5 a) Show that $2\mathbb{Z} \cup 5\mathbb{Z}$ is not a subring of $\mathbb{Z}$

Not closed under addition

$$2 + 5 = 7 \notin 2\mathbb{Z} \cup 5\mathbb{Z}$$

b) Show that $2\mathbb{Z} + 5\mathbb{Z} = \{2n + 5m \mid n, m \in \mathbb{Z}\} = \mathbb{Z}$

It is a subring of $\mathbb{Z}$ containing all of $\mathbb{Z}$, since $\gcd(2,5) = 1$, so all integers are of the form $\{2n + 5m \mid n, m \in \mathbb{Z}\}$

c) Show that $2\mathbb{Z} \cap 5\mathbb{Z} = 10\mathbb{Z}$

$x \in 2\mathbb{Z} \cap 5\mathbb{Z}$, then $2 \mid x$ and $5 \mid x$

$\Rightarrow \operatorname{lcm}(2,5) \mid x$ i.e. $10 \mid x$

$\Rightarrow x \in 10\mathbb{Z}$

$x \in 10\mathbb{Z} \Rightarrow x \in 2\mathbb{Z}$ and $x \in 10\mathbb{Z}$

So $2\mathbb{Z} \cap 5\mathbb{Z} = 10\mathbb{Z}$

5.2.6 $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$

Prove or disprove R is a subring of the ring of all 2×2 integer matrices under componentwise addition and matrix multiplication

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Subring test 5.2.2

Check closed under subtraction OK

———— " ———— multiplication

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix} \quad \text{OK}$$

So it is a subring