5.2.7 Prove that $R[x]$ is a commutative ring with unity when R is a commutative ring with unity

$f(x) = a_0 + a_1 x + \cdots + a_n x^{n_1}$

$g(x) = b_0 + b_1 x + \cdots + b_n x^{n_2}$

$h(x) = c_0 + c_1 x + \cdots + c_n x^{n_3}$

WLOG $n_1 = n_2 = n_3 = \max(n_1, n_2, n_3)$ by adding terms with zero coefficients

$(f+g)(x) = a_0 + b_0 + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$

$(f+g)+h = (a_0+b_0)+c_0 + ((a_1+b_1)+c_1)x + \cdots + ((a_n+b_n)+c_n)x^n$

We see that $f+g = g+f$, so $R[x]$ is an abelian group under $+$

$$fg(x) = \sum_{k=0}^{n}\left(\sum_{i+j=k} a_i b_j\right)x^k$$

$$(fg)h = \sum_{k=0}^{2n}\left(\sum_{i+j=k} a_i b_j\right)x^k \cdot \sum_{l=0}^{n} c_l x^l$$

$$= \sum_{\alpha=0}^{3n}\left(\sum_{k+l=\alpha}\left(\sum_{i+j=k} a_i b_j\right)c_l\right)x^\alpha$$

$$= \sum_{\alpha=0}^{3n}\sum_{i+j+l=\alpha} a_i b_j c_l \, x^\alpha$$

So we see that $(R, \cdot)$ is also associative and commutative.

Can also check that the distributive property holds and 1 is a unit

5.2.9 Find the group of units in $\mathbb{R}[x]$

Assume $f(x)\,g(x) = 1$

Then $0 = \deg(fg) = \deg f + \deg g$

$\Rightarrow \deg f = \deg g = 0$ Since $\deg f \geq 0$

So f and g are units in $\mathbb{R}$, i.e. any nonzero real number

5.2.11 Find the units in the ring $\mathbb{Z}^{2\times 2}$ of 2×2 matrices and the usual matrix multiplication

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if A^{-1} is defined

So the units are matrices A s.t. $\det(A) = \pm 1$

Since these are the units in $\mathbb{Z}$

5.2.14 Show that the Lie bracket

$$[A,B] = AB - BA \quad \text{makes} \quad (M, +, [\cdot,\cdot])$$

into a non-associative ring satisfying the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

$$[A,[B,C]] = A(BC-CB) - (BC-CB)A = ABC - ACB - BCA + CBA$$

$$[[A,B],C] = (AB-BA)C - C(AB-BA) = ABC - BAC - CAB + CBA$$

These are different, so not associative

$$[C,[A,B]] = -[[A,B],C]$$

$$[B,[C,A]] = BCA - BAC - CAB + ACB$$

Recognize that Jacobi identity holds by cancelling terms

5.2.16 Show that $\mathbb{Z}[[X]]$ is a commutative ring with unity and $f \cdot g = 0 \Rightarrow f = 0$ or $g = 0$

We see from the definition that $fg = gf$

$$f = \sum_{n=0}^{\infty} a_n x^n \qquad g = \sum_{n=0}^{\infty} b_n x^n$$

$$fg = \sum_{k=0}^{\infty} \Big(\sum_{\substack{m+n=k \\ 0 \leq m,n \leq k}} a_m b_n \Big) x^k$$

Assume $fg = 0$, then

$$\sum_{\substack{m+n=k \\ 0 \leq m,n = k}} a_m b_n = 0 \qquad \forall k \in \mathbb{N}$$

$a_0 b_0 = 0 \Rightarrow a_0$ or $b_0 = 0$ Assume $a_0 = 0$ and $a_i \neq 0$ $i > 0$ $b_j \neq 0$

Assume i is the lowest non-zero coefficient of f and j the lowest non-zero coefficient of g

$$\sum_{\substack{m+n=i+j \\ 0 \leq m,n \leq i+j}} a_m b_n = 0 \qquad a_i b_j = -\sum_{\substack{m+n=i+j \\ 0 \leq m,n \leq i+j \\ (m,n) \neq (i,j)}} a_m b_n$$

By assumption the left side is non-zero and the right side is zero. Therefore either f or g must be zero.

5.3.6. ① Nothing to prove

② —— " ——

③ —— " ——

④ $\mathbb{Z}_p[x]$ has characteristic p since

$$\underbrace{1+\dots+1}_{p\text{-times}} = 0$$

It has infinitely many elements since it contains the elements $\{x^i\}_{i \in \mathbb{N}}$

5.3.7 Prove that $\mathbb{Z}_p(0)$ and $R = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{Z}_p \right\}$

are isomorphic.

$$F : \mathbb{Z}_p(0) \to R$$

$$x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

F is a ring homomorphism since

$$F(x+y) = \begin{pmatrix} 0 & x+y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = F(x) + F(y)$$

$$F(xy) = F(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = F(x)\,F(y)$$

F is bijective.

5.3.9. Which of these rings are integral domains and which are fields

① $\mathbb{Z}[i]$ is an integral domain since it is a subring of the integral domain $\mathbb{C}$, but not a field since 2 does not have an inverse

② $\mathbb{Z}/_{12\mathbb{Z}}$ is not an integral domain since

$3 \cdot 4 = 0$

③ $\mathbb{Z}_2^{2\times 2}$ is not an integral domain since

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

④ $\mathbb{Z}_{11}$ is a field since 11 is a prime

⑤ $\mathbb{Z} \oplus \mathbb{Z}$ is not an integral domain since

$(1,0) \cdot (0,1) = (0,0)$

⑥ $\mathbb{Q}$ is a field

⑦ $R[x]$ is not in general an integral domain, only when R is

5.4.4 Show that $\mathbb{Q}[x]/(x^2-2)$

is isomorphic to $\mathbb{Q}[\sqrt{2}]$

We will show that $[x] \in \mathbb{Q}[x]/(x^2-2)$

acts as $\sqrt{2}$

$$[x]^2 = [x^2] = [x^2 - (x^2-2)] = [2]$$

More formal way of proving this

Show that for $F: \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$

defined by $F(1)=1$ $F(x)=\sqrt{2}$

has kernel (x^2-2) and use an isomorphism theorem

5.4.14 Is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ a field or integral domain?
What about $\mathbb{Z}_2 \oplus \mathbb{Z}_3$?

No, neither ring is an integral domain since

$(1,0)\cdot(0,1) = (0,0)$

5.4.15 Find the characteristics of
a) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ b) $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ c) $\mathbb{Z}_2 \oplus \mathbb{Z}_3$

a) $2(1,1) = (0,0)$ char is 2

b) $4(1,1) = (0,0)$ char is 4

c) $6(1,1) = (0,0)$ char is 6

5.4.16 Find a subring of $R = \mathbb{Z} \oplus \mathbb{Z}$ that is not an ideal

Following the hint we look at $S = \{(a,b) \mid a+b \text{ is even}\}$

S is a subring, since it is closed under subtraction and multiplication

S is not an ideal since

$(3,5) \in S$ but $(1,0)(3,5) = (3,0) \notin S$