Suggested solution to the assignment in MAT 2200 / 2021 (1)

(1)

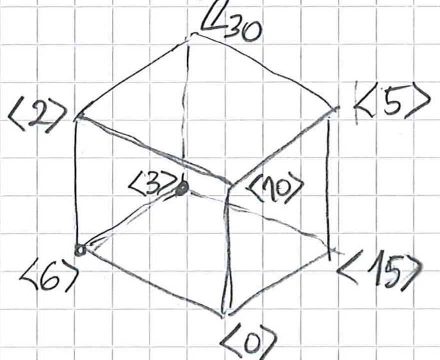
(a) $\mathbb{Z}_{30}$. The prime divisors of 30 are 2, 3 and 5.

$\langle k\rangle$ has order $\dfrac{30}{\gcd(30,k)}$ in $\mathbb{Z}_{30}$

$|\langle 2\rangle|$ has order 15 with subgroups $\langle 6\rangle$ of order 5 & $\langle 10\rangle$ of order 3

$|\langle 3\rangle|$ — " — 10 — " — $\langle 6\rangle$ of order 5 & $\langle 15\rangle$ of order 2

$|\langle 5\rangle|$ has order 6 — " — $\langle 10\rangle$ of order 3 and $\langle 15\rangle$ of order 2.



(b) # Elem's of order 2, 3, 4 in $\mathbb{Z}_{30}\oplus\mathbb{Z}_2$.

$|(a,b)| = \mathrm{lcm}(|a|,|b|)$, $|a|$ divisor of 30, $|b|$ divisor of 2,

so $|a| \in \{1,2,3,5,2\cdot 3, 2\cdot 5, 3\cdot 5\}$ and $|b|\in\{1,2\}$

$\mathrm{lcm}(|a|,|b|)=2 \iff |a|=2, |b|=1$, $|a|=1, |b|=2$ or $|a|=|b|=2$.

(any of the factors 3 or 5 in $|a|$ will appear in the lcm)

$|a|=2 \iff a=15$ in $\mathbb{Z}_3$ so $(15,0)$, $(0,1)$ or $(15,1)$.

$\mathrm{lcm}(|a|,|b|)=3 \iff |a|=3, |b|=1$.

$a=10$ and $a=20$ have order 3 in $\mathbb{Z}_{30}$.

$\Rightarrow (10,1), (20,1)$

$\mathrm{lcm}(|a|,|b|)=4$ requires $|a|=4$ in $\mathbb{Z}_{30}$, but 4 does not divide 30, so there are no elements of order 4 in $\mathbb{Z}_{30}$. Hence there are no elements of order 4 in $\mathbb{Z}_{30}\oplus\mathbb{Z}_2$.

① (c) Draw the poset diagram for subgroups of $\mathbb{Z}_{15}^*$ under ·.

Have $\mathbb{Z}_{15}^* = \{a \in \mathbb{Z}^+, a \leq 15, \gcd(a,15)=1\}$

so $\mathbb{Z}_{15}^* = \{1,2,4,7,8,11,13,14\}$. Thus $\mathbb{Z}_{15}^*$ has order 8.
Since multiplication mod 15 is commutative, we have that

$\mathbb{Z}_{15}^*$ is an abelian group of order 8. By lecture 16, $\mathbb{Z}_{15}^*$ is isomorphic to exactly one of C_8, $C_4 \oplus C_2$ or $C_2 \oplus C_2 \oplus C_2$.
We compute the order of elements in $\mathbb{Z}_{15}^*$ and get

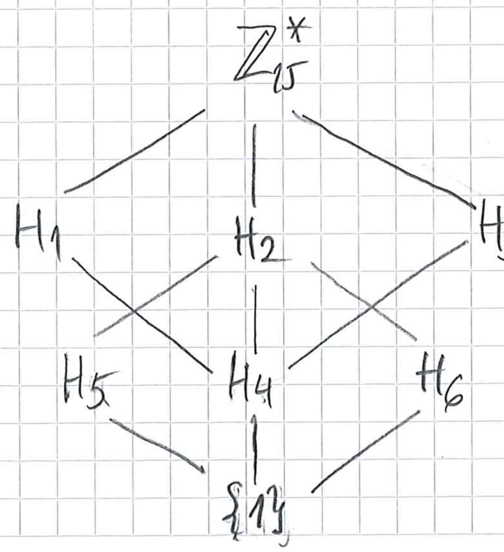
$|2| = 4$ (since $2^4 \pmod{15} \equiv 16 \pmod{15} \equiv 1 \pmod{15}$). Similarly, 7, 8 and 13 have order 4.
The elements 4, 11 and 14 have order 2. The identity has order 1.
Since no element has order 8 and there are elements of order 4, we see that $\mathbb{Z}_{15}^* \cong C_4 \oplus C_2$ (the largest order of elements in $C_2 \oplus C_2 \oplus C_2$ is $\operatorname{lcm}(2,2,2)=2$). We get the subgroups.

$H_1 = \{1,2,4,8\}$ is cyclic of order 4 (so $H_1 \cong C_4 \oplus \{0\}$).

$H_2 = \{1,4,11,14\}$ contains 3 elements of order 2, so $H_2 \cong V = C_2 \oplus C_2$

$H_3 = \{1,7,4,13\}$ is cyclic of order 4. (so $H_2 \cong \langle(1,1)\rangle \cong C_4$.

$H_4 = \{1,4\}$, $H_5 = \{1,11\}$, $H_6 = \{1,14\}$ have order 2.

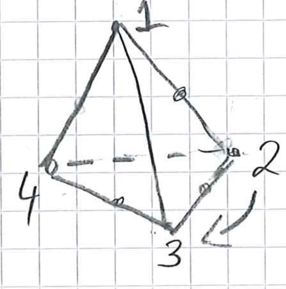


2 G group, H finite subgroup of order $|H| = n$, $n \in \mathbb{Z}^+$.
If H is unique such, show it is normal in G.

Solution: Let $g \in G$, consider the conjugation isomorphism
$C_{g^{-1}}: G \to G$, $C_{g^{-1}}(x) = gxg^{-1}$ for $x \in G$.
Then, $C_{g^{-1}}(H)$ is a subgroup of G, also denoted gHg^{-1}.
Since $C_{g^{-1}}$ is bijective from H onto gHg^{-1}, we have $|gHg^{-1}| = n$.
By our hypothesis, we must have $gHg^{-1} = H$. Since g was arbitrary, this shows H is normal.

3 S_4 = symmetric group on $\{1, 2, 3, 4\}$, A_4 the alternating group.

(a) Identify A_4 as symmetries of the regular tetrahedron and show A_4 is generated by $R = (234)$ and $F = (12)(34)$ with relations $R^3 = F^2 = (FR)^3 = I$.



Choose $I \in S_4$ to represent the tetrahedron (other choices of the labelling of the bottom triangle are possible).

Rotation by $2\pi/3$ about axis through 1 that is perpendicular to the bottom triangle is given by: (counterclockwise)

$$R = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (2\,3\,4)$$

Rotation by 2π about axis connecting the middle points of the segments (12) and (34) is given by the flip

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\,2)(3\,4).$$

We easily see that $R^2 = (432)$ and $R^3 = I = F^2$.

3(a) continued.

Similarly to R we obtain rotations about axes through the vertices 2, 3 and 4:

$R_2 = (134)$; $R_3 = (124)$, $R_4 = (123)$. Clearly, $R_3^3 = I = R_4^2$.

Similar to F we obtain two more flips $(13)(24)$ and $(14)(23)$.

To show that F and R generate A_4, we compute products of F, R and R^2 and check that we get all 12 elements of A_4. We already have 4 elements: I, R, R^2, F. Then we compute:

$FR = R_3$, $RF = R_4^2$, $FR^2 = R_4$, $R^2F = R_3^2$, $RFR = R_2$, $FRF = R_2^2$,

and we get 6 more elements (all cycles of length 3).

The remaining 2 elements are obtained as

$R_3R_4 = (14)(23)$ and $RR_2 = (13)(24)$, thus are also expressed as products of R, F and R^2. Since FR is the rotation R_3, we have $(FR)^3 = I$, as claimed. In conclusion, we can write

$$A_4 = \{I, R, R^2, F, FR, R^2F, FR^2, RF, RFR, FRF, FRFR^2, R^2FR\}.$$

(b) The conjugacy classes of A_4:

$\mathcal{C}_1$ = the class of I

$\mathcal{C}_2$ = the class of F, also containing $(13)(24)$ and $(14)(23)$.

$\mathcal{C}_3$ = the class of R: note that R and R^2 are not in the same conjugacy class, otherwise for some $\sigma \in A_4$ we have that $\sigma R \sigma^{-1} = (\sigma(2)\,\sigma(3)\,\sigma(4))$ must equal $R^2 = (243)$. This forces $\sigma = (34)$, which is an odd permutation in S_4, and thus cannot be in A_4. Similarly, R_2 and R_2^2 are not in the same class, nor are R_3, R_3^2 or R_4, R_4^2.

Now, $FRF^{-1} = FRF = R_2^2$ implies $R_2^2 \in \mathcal{C}_3$,

$RF\cdot R\,(RF)^{-1} = RF\cdot R\cdot FR^2 = (124) = R_3$ implies $R_3 \in \mathcal{C}_3$,

and $R^2F\cdot R\cdot (R^2F)^{-1} = R^2F\cdot RFR = R^2\cdot (FR)^{-1} = R^2\cdot R^{-1}F^{-1}$

$= R^2R^2F = RF = R_4^2$ implies $R_4^2 \in \mathcal{C}_3$.

Thus $|\mathcal{C}_3| \geq 4$, however $|\mathcal{C}_3|$ divides $|A_4| = 12$ and cannot contain any more 3-cycles (by the previous observation), so $|\mathcal{C}_3| = 4$ and $\mathcal{C}_3 = \{R, R_2^2, R_3, R_4^2\} = \{(234), (143), (124), (132)\}$

Finally, $\mathcal{C}_4$ = the class of R^2, must also consist of only cycles and satisfy $|\mathcal{C}_4|$ divides 12, and since it contains also least R_2, R_3^2 and R_4, it is exactly $\{(243), (134), (142), (123)\}$.

Since $Z(A_4) = \{I\}$, the class equation is $12 = 1+3+4+4$.

(c) Show A_4 has a normal subgroup of order 4 and no subgroup of order 6.

For $H \subset A_4$ to be a subgroup need $I \in H$. For H to be normal, we need $\sigma H \sigma^{-1} \in H$, $\forall \sigma \in A_4$. Thus, if $F \in H$, then $\mathcal{C}_2 \subset H$.

If H contains any σ in $\mathcal{C}_3$ or in $\mathcal{C}_4$, it contains the entire class of 4 cycles of length 3, but for it to be a subgroup it must contain their inverses, too, so we would get $|H| \geq 8$, contradicting Lagranges thm. We see that

$H = \{I, (12)(34), (13)(24), (14)(23)\}$ is a subgroup of A_4 of order 4, and it is the unique such, as it consists of I and a conjugacy class of order 3. By Problem 2, H is normal in A_4.

Assume $K \subset A_4$ is a subgroup of order $|K| = 6$, then the index $[A_4 : K] = \frac{|A_4|}{|K|}$ is 2, therefore K would be normal in A_4 (by a result in the course). But the only normal subgp we have in A_4 is $H = \mathcal{C}_1 \cup \mathcal{C}_2$ of order 4, and $K = H$ is a falsehood.

4 $G = \mathbb{Z}_{36}$, $H = \langle 18 \rangle$, $K = \langle 9 \rangle$

(a) There are 18 cosets $x + H$ in G/H, $0 \le x \le 17$. (List them.)
There are 9 cosets $x + K$ in G/K, $0 \le x \le 8$. (List them.)
There are 2 cosets in K/H : $0 + H = \{0, 18\}$, $9 + H = K \setminus H = \{9, 27\}$

(b) The bijection $T: (G/H)/(K/H) \longrightarrow G/K$ takes

$$(x+H) + K/H \text{ to } x+K \quad , \quad 0 \le x \le 8$$

$$T((0+H) + K/H) = T(\{\{0,18\}, \{9,27\}\}) = 0 + K$$
$$T((1+H) + K/H) = T(\{\{1,19\}, \{10,28\}\}) = T(1+H, 10+H) = 1+K.$$
$$T((2+H) + K/H) = T(\{2,20\}, \{11,29\}) = T(2+H, 11+H) = 2+K$$
$$\vdots \qquad \qquad \vdots$$
$$T((8+H) + K/H) = T(\{8,26\}, \{17,35\}) = T(8+H, 17+H) = 8+K.$$

5 (a) G group, A, B normal subgroups of finite index.

Solution (other solutions are possible).
We have also $A \cap B$ normal in G and $A \cap B$ normal in B
By the 3rd Isom thm, $\dfrac{|G/A\cap B|}{|B/A \cap B|} = |G/B| = (G:B)$,
thus $(G:B)\, |B/A\cap B| = |G/A \cap B| = (G : A\cap B)$. To finish, it suffices to show that $|B/A\cap B| \le (G:A)$. The map $f: B/A\cap B \to G/A$, $f(x(A\cap B)) = xA$ is well-defined and injective (Check!), so $(B : A \cap B) \le (G:A)$, as wanted.

(b) Show that for arbitrary subgroups A, B of G of finite index $(G:A)$, $(G:B)$ respectively, we have

$$(G:A\cap B) \leq (G:A)(G:B)$$

It suffices to show that $(G:B)(B:A\cap B) \overset{\circledast}{=} (G:A\cap B)$ is valid only assuming that $(G:A)$, $(G:B)$ are finite, in which case it follows that $(G:A\cap B)$ is finite. The map $f(x(A\cap B)) = xA$, $x\in B$ makes sense for coset spaces $B/_{A\cap B}$, $G/_A$, and does not depend on these being groups.

The general form of $\circledast$ is: for K, H subgroups of G with $K\subset H$ and $(G:H)$, $(H:K)$ finite (thus the left coset spaces $G/_H$, $H/_K$ are finite) we have $(G:H)(H:K) = (G:K)$, and the index in the right-hand side is finite, too.

For $\{a_iH \mid 1\leq i\leq (G:H)\}$ and $\{b_jK \mid 1\leq j\leq (H:K)\}$ the distinct cosets,
show that $\{a_ib_jK \mid 1\leq i\leq (G:H), 1\leq j\leq (H:K)\}$ are distinct cosets.