Group actions and Burnside's Thm § 16 + 17

Recall:

Def 16.1 Let X be a set and G be a group

An action of G on X is a map $*: G \times X \to X$ s.t.

1. $e * x = x \quad \forall\ x \in X$
2. $(g_1 g_2) * x = g_1 * (g_2 * x) \quad \forall\ x \in X$ and $g_1, g_2 \in G$

Main example: S_n acts on $\{1, \ldots, n\}$

Isotropy subgroups

Def Let X be a G-set. Define

$$X_g = \{x \in X \mid gx = x\} \subseteq X \quad \text{and} \quad G_{x \in X} = \{g \in G \mid gx = x\}.$$

"elements of X fixed by $g \in G$"

stabiliser or isotropy subgroup

"elements of G fixing x"

Examples

$X = \{1, \dots, n\}$ $\quad G = S_n$

$\tau = (1,2)$

$X_\tau = \{3, 4, 5, \dots n\}$

$$G_{1 \in X} = \{e, (2,3)(4,5,6), \dots\}$$
$$= S_{X \setminus \{1\}}$$
$$\cong S_{n-1}$$

Thm 16.12 G_x is a subgroup of G

Proof 1) closed: $g_1, g_2 \in G_x$ then

$$(g_1 g_2)x = g_1(g_2 x) = g_1 x = x \Rightarrow g_1 g_2 \in G_x$$

since $g_2 \in G$ $g_2 x = x$

2) $e \in G_x$ since

$ex = x$

3) inverses in G_x

$g \in G_x$ then

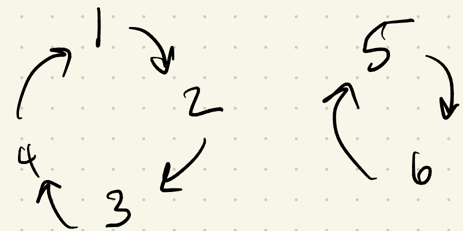
$$x = ex = (g^{-1}g)x = g^{-1}(gx) = g^{-1}x$$

$\Rightarrow g^{-1}x = x$ $g^{-1} \in G_x$

$\square$

Orbits. Recall $\sigma \in S_n$ we considered its cycles.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$$



$\to (1,2,3,4)(5,6)$.

$G = \langle \sigma \rangle$ acting on $X = \{1, \dots, 6\}$. has orbits $\{1,2,3,4\}$ $\{5,6\}$

Thm 16.14 Let X be a G-set. For $x_1, x_2 \in X$ define $x_1 \sim x_2 \iff \exists\, g \in G$ s.t. $gx_1 = x_2$.

This is an equivalence relation on X.

Proof See text. Reflexive $(e x_1 = x_1)$ Symmetric $(g^{-1} x_2 = x_1)$

Transitive $(g_1 x_1 = x_2 \quad g_2 x_2 = x_3 \quad (g_2 g_1) x_1 = x_3$.

Def Let X be a G-set. A cell in the partition of the equivalence relation is an orbit in X under G.

$$\mathcal{O}_x = \{x' \in X \mid gx = x' \; g \in G\}.$$

$\mathcal{O}_{x'}''$

⚠ Book writes GX. but gets confused with G_x

Example: If G acts transitively then there is only one orbit $\mathcal{O}_x = X$

- Let $G \leq \overset{X}{\underset{\|}{G'}}$ define an action of G on G' by
$$G \times G' \to G'$$
$$(g, g') \mapsto gg'$$
What are the orbits? The right cosets $\{Gg'\}$ of G in G'.

Thm 16.16 Let X be a G-set. Then

$$|\mathcal{O}_x| = (G : G_x) \leftarrow \text{index} = \#\text{ of cosets of } G_x \text{ in } G.$$

Proof. Define $\psi : \{G_x, g_1G_x, g_2G_x, \ldots\} \to \mathcal{O}_x$

$$\psi(gG_x) = gx$$

Proceed with caution: Is ψ well defined?

suppose $gG_x = g'G_x$ then $\psi(gG_x) = gx$

$\Rightarrow g' = g\hat{g}$ $\quad\psi(g'G_x) = (g\hat{g})x$

$\hat{g} \in G_x$ $\quad = g(\hat{g}x) = gx$

$\Rightarrow \psi$ well defined

Define $\psi : \{G_x, g_1 G_x, g_2 G_x, \dots\} \to \mathcal{O}_x$ $\quad \psi(g G_x) = gx$
2

<u>ψ is one to one:</u> Suppose cosets gG_x and $g' G_x$ satisfy

$$\psi(gG_x) = \psi(g' G_x) \Rightarrow gx = g'x$$

$$x = (g^{-1} g')x$$

$$\Rightarrow g^{-1} g' \in G_x \Rightarrow g' \in g G_x$$

$$\Rightarrow g' G_x = g G_x$$

$\Rightarrow \psi$ one to one

<u>ψ is onto:</u>

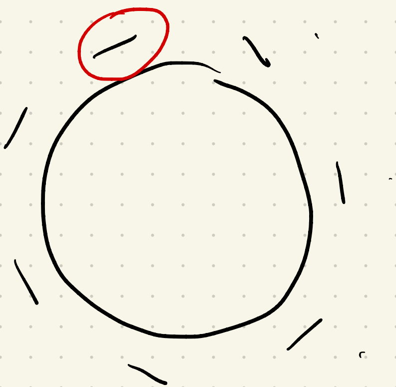
Suppose $x_1 \in \mathcal{O}_x \Rightarrow gx = x_1$ then

$$\psi(gG_x) = gx = x_1 \quad \Rightarrow \psi \text{ onto}$$

$\square$

G-sets and Counting with symmetries §17

Ex 17.4 How many ways are there to seat n people around a circular table?



1) if the table has a distinguished seat $n(n-1)\cdots 2\cdot 1 = n!$

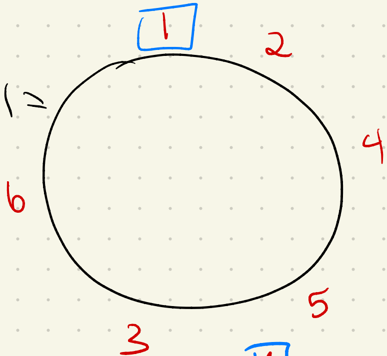
2) if no distinguished seat there are $\dfrac{n!}{n} = (n-1)!$

$n=7$ $\quad X = \{$seating arrangements with a distinguished seat$\}$ $\quad |X| = n!$

$G = \mathbb{Z}_n$ acts on X $\quad ix = x'$ move everyone i spots to left. $\quad i \in \mathbb{Z}_n$, $x \in X$

Answer to part 2) is # of orbits of X under G.

$x_1 =$

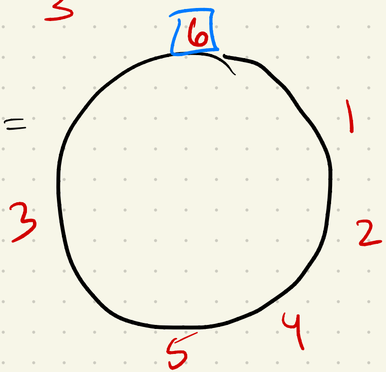


People $= \{1, \ldots, n\}$

$X = \{$ arrangements with a distinguished seat $\}$

$x_1 \neq x_2 \in X$.

$1 \cdot x_1 =$

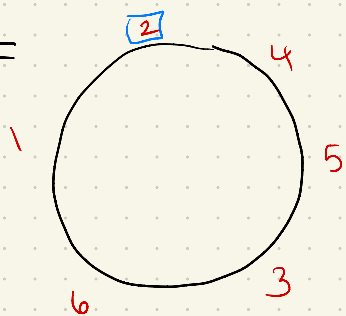


$\langle 1 \rangle = \mathbb{Z}/n = G$

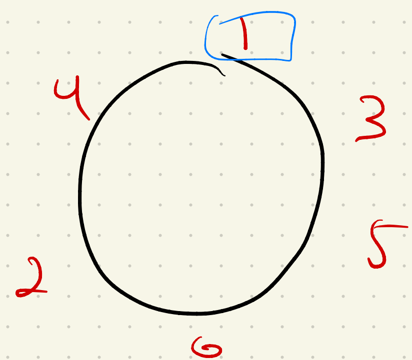
But removing the distinguished seat they become the same

$X_2 =$

$5 \cdot X_1$



x_3



seating arrangements is # of orbits of X under G.

What if we don't distinguish between clockwise or counter-clockwise arrangements? Ie. Beaded necklaces?

Thm 17.1 (Burnside's formula) Let G be a finite gp and X a finite G-set. Then

$$\#\left\{\begin{matrix}\text{orbits of} \\ X \text{ under } G\end{matrix}\right\} = \frac{1}{|G|} \sum_{g \in G} |X_g|$$

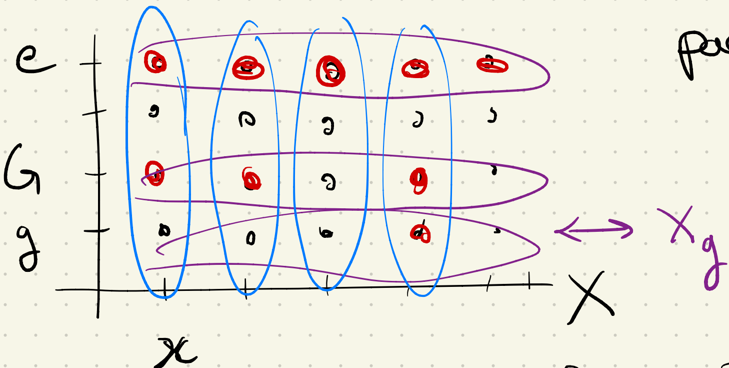
$$X_g = \{ x \in X \mid gx = x \}.$$

$X_g \subseteq X$ ⚠ Do not partition

eg. $X_e = X$

Proof By "double counting" (Mat2250)

$$\mathcal{N} = \{ (g, x) \mid gx = x \} \subseteq G \times X$$



partition $\mathcal{N}$ by the horizontal rows

$$|\mathcal{N}| = \sum_{g \in G} |X_g| \qquad X_g \updownarrow$$

Since Fixing $g' \in G$ then a row is $\{(g', x) \mid g'x = x\}$

Partition along columns x' the column is $\{(g, x') \mid gx' = x'\} \leftrightarrow G_x$

$$|\mathcal{O}_x| = (G : G_x)$$

$$|\mathcal{N}| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|\mathcal{O}_x|} = |G| \sum_{x \in X} \frac{1}{|\mathcal{O}_x|}$$

If $x' \in \mathcal{O}_x$ then $\mathcal{O}_x = \mathcal{O}_{x'} \Longrightarrow |\mathcal{O}_x| = |\mathcal{O}_{x'}|$

Denoting an orbit by $\mathcal{O}$: $\displaystyle\sum_{x \in \mathcal{O}} \frac{1}{|\mathcal{O}|} = \frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} 1 = \frac{|\mathcal{O}|}{|\mathcal{O}|} = 1$

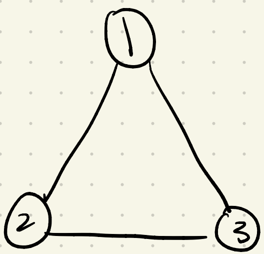
$$\Rightarrow \sum_{x \in X} \frac{1}{|\mathcal{O}_x|} = \text{\# orbits of } X \text{ under } G.$$

$$\#\begin{Bmatrix}\text{orbits of} \\ X \text{ under } G\end{Bmatrix} |G| = |N| = \sum_{g \in G} |X_g|.$$

$\square$

Example 17.6

of ways of coloring the vertices equilateral triangle with 4-colors with repetitions allowed.



$$X = \{ \text{colorings of labelled triangle, } 4 \text{ colours + repetitions} \}$$

$$|X| = 64.$$

$G = S_3$

want to count # of orbits.

Example 17.6

\# of ways of coloring the vertices equilateral triangle with 4-colours with repetitions allowed?

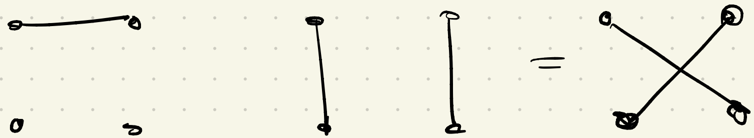
Apply Bernstains Thm: $\left\{ \begin{matrix} \#\text{orbits of} \\ X \text{ under } G \end{matrix} \right\} \cdot |G| = \sum_{g \in G} |X_g|$

Challenge Mat 2250.

ways of connecting dots.

How many indistinguishable graphs are there on 4 vertices?

eg.



$X \stackrel{?}{=}$

$G \stackrel{?}{=}$