Group Theory Wrap-up

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1^{st} Sylow: There exists a p-Sylow subgp.

2^{nd} Sylow: All p-Sylow subgp are conjugate.

3^{rd} Sylow: $n_p \equiv 1 \pmod{p}$, n_p divides $|G|$.

Lemma: If $H, K \triangleleft G$, $H \cap K = \{e\}$,
$H \vee K = G$, then $G \cong H \times K$.

join

$H \vee K$ smallest subgp $\geq H, K$

Prop: If $p < q$ are prime s.t. $q \not\equiv 1 \pmod{p}$,
then $|G| = pq \Rightarrow G \cong \mathbb{Z}_{pq}$.

Remark: $S_3 = 2 \times 3$ is not abelian

Prop: If $|G| = p^2$, then G is
abelian, i.e., $G \cong \mathbb{Z}_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}_p$.

Proof:

Case 1: G has an element g of order p^2
$G = \langle g \rangle \cong \mathbb{Z}_{p^2}$, done.

Case 2: Pick $a \in G$ not identity

a's order is p. $\langle a \rangle$ is $\cong \mathbb{Z}_p$

Pick $b \notin \langle a \rangle$, b's order is p.

$\langle b \rangle$ is $\cong \mathbb{Z}_p$ (not equal to $\langle a \rangle$)

(1) $\langle a \rangle \cap \langle b \rangle$ is $\{e\}$.

Suppose $\exists\, c \in \langle a \rangle \cap \langle b \rangle$ not e.

Then c generates both $\langle a \rangle$ and $\langle b \rangle$.

So $\langle a \rangle = \langle c \rangle = \langle b \rangle$, contradiction

(2) $\langle a \rangle \vee \langle b \rangle = G$.

$|\langle a \rangle \vee \langle b \rangle| > |\langle a \rangle| = p$

but $|\langle a \rangle \vee \langle b \rangle|$ divides $|G| = p^2$.

$\therefore |\langle a \rangle \vee \langle b \rangle| = p^2$

By lemma: $G \cong \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$. $\square$

Remark: $|D_4| = 8 = 2^3$ is not abelian.

Motto I: Many maths objects either (i) have a group structure, or (ii) have groups acting on them, or (iii) have an invariant that is a group.

Motto II: Many questions about groups can be asked for other objects.

For groups, we asked what are

- homomorphisms, isomorphisms
- subgroups, quotient groups
- (direct) products
- "Standard" objects: cyclic groups and symmetric groups, etc.

Ways to understand groups:

(1) Make G into a subgp/quotient group of "standard" objects.

- Cayley thm: Every finite gp is a subgp of some S_n.
- Group representation:
  $G \longrightarrow \{\text{Matrices}\}$

(2) Understand "pieces" of G and how to "paste" them together.

- Fundamental thm f.g. abelian gp (Direct produce of cyclic groups)
- Sylow thm, solvable groups.

(3) Understand how G acts on another set X.
(or understand X from G)

- Burnside's formula: find the # of objects up to symmetry.
- Geometric group theory: Understand G by understanding $G \curvearrowright$ geometric obj.
- Galois theory: groups acting on fields.