Fields of Fractions (I)

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Recall: $\mathbb{Z}$ is an integral domain, so it has cancellation law.

$$ab = ac,\ a \neq 0 \Rightarrow b = c.$$

But $\mathbb{Z}$ is not field.

Still, we have $\mathbb{Q}$

Goal: For any integral domain D, construct a field K s.t. $D \subseteq K$.

$2x = 2y \qquad 2 \in \mathbb{Z}$

$\left(\frac{1}{2}\right)2x = \left(\frac{1}{2}\right)2y \qquad 2 \in \mathbb{Q}$

$\qquad \frac{1}{2} \in \mathbb{Q}$

$x = y$

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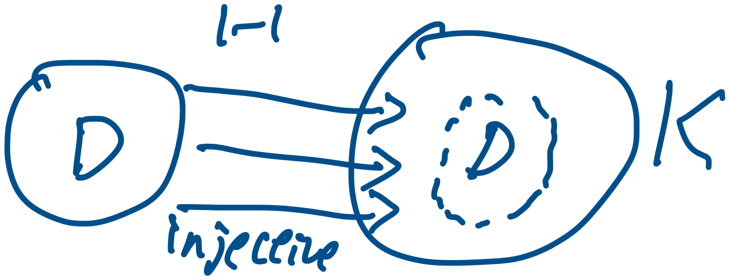
Construct $(K, +, \cdot, 0, 1)$ and identify D inside K



$$\mathbb{Q} = \left\{ \frac{n}{m} : n, m \in \mathbb{Z}, m \neq 0 \right\}.$$

$$= \Big\{ \underbrace{\left\{ \frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots \right\}}_{} {}^{=0},$$

$$\left\{ \frac{1}{1}, \frac{2}{2}, \frac{-3}{-3}, \ldots \right\}^{=1},$$

$$\left\{ \frac{1}{2}, \frac{2}{4}, \frac{-3}{-6}, \frac{3}{6}, \ldots \right\}^{=1/2},$$

$$\ldots \Big\}$$

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

$$\tilde{K} = \{ (a, b) : a, b \in D, b \neq 0 \}.$$

Define $\sim$ on $\tilde{K}$ by

$$(a, b) \sim (c, d) \iff ad = bc.$$

$K = \tilde{K}/\sim$

Check $\sim$ is an equivalent relation.

Reflexive ($x \sim x$):

$(a,b) \sim (a,b)$ b/c $ab = ba$

Symmetric ($x \sim y \Rightarrow y \sim x$):

$$(a,b) \sim (c,d) \Rightarrow ad = bc$$
$$\Rightarrow cb = da$$
$$\Rightarrow (c,d) \sim (a,b)$$

Transitive ($x \sim y, y \sim z \Rightarrow x \sim z$):

$$\begin{matrix} (a,b) \sim (c,d) \\ (c,d) \sim (e,f) \end{matrix} \Rightarrow \begin{matrix} ad = bc \\ cf = de \end{matrix}$$

$$\underline{a}d f = \underline{b c f} = b\underline{de}$$

$$\overset{CP}{\Rightarrow} af = be \Rightarrow (a,b) \sim (e,f).$$

$$[(a,b)] +_K [(c,d)]$$
$$= [(ad +_D bc, bd)]$$
$$[(a,b)] \cdot_K [(c,d)]$$
$$= [(ac, bd)].$$