Fields of Fractions (II)

Thursday, 10 March 2022 12:10

D integral domain $\mathbb{Z} \Rightarrow \mathbb{Q}$

$$K = \{(a,b) : a,b \in D, b \neq 0\}/\sim$$

$$(a,b) \sim (c,d) \Leftrightarrow ad = bc$$

$$[(a,b)] +_K [(c,d)] = [(ad +_D bc, bd)]$$

$$[(a,b)] \cdot_K [(c,d)] = [(a \cdot_D c, b \cdot_D d)]$$

(1) $\sim$ is an equiv. relation ✓ (Yesterday)

(2) $+_K$ is well-defined ✓

(3) $+_K$ is associative ✓ } Exer

(4) $+_K$ is commutative ✓ }

(5) $[(0,1)]$ is the additive identity ✓

(6) $[(-a,b)]$ is the add. inverse of $[(a,b)]$ ✓

(7) $\cdot_K$ is well-defined ✓ Similar to (2)

(8) $\cdot_K$ is associative ✓ } Exer

(9) $\cdot_K$ is commutative ✓ }

(10) $[(1,1)]$ is the multiplicative identity ✓

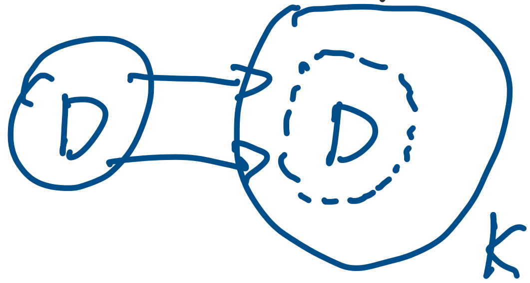
(11) $[(b,a)]$ is the multi. inverse of $[(a,b)] \neq 0_K$ ✓

(12) Distributive law holds ✓ Exer

(13) There is an injective homomorphism

$\iota : D \to K$ ✓

(Pretend $D \subseteq K$)



Check (2): $+_K$ is well-defined.

$$[(a,b)] +_K [(c,d)] = [(ad+bc, bd)]$$

Suppose I chose $(a',b') \in [(a,b)]$.

i.e. $a'b = ab'$

$$[(a',b')] +_K [(c,d)] = [(a'd+b'c, b'd)]$$

Want to show

$$(ad+bc, bd) \sim (a'd+b'c, b'd)$$

$$\iff adb'd + bcb'd = bda'd + bdb'c$$

(5) $[(0,1)]$ is the additive identity.

$$[(0,1)] = \{(0,r) : r \in D, r \neq 0\}. \ (*)$$

$(a,b) \in [(0,1)] \iff a \cdot 1 = b \cdot 0 = 0$

$\iff a = 0$

Now for any $[(a,b)] \in K$,

$$[(0,1)] + [(a,b)] = [(0\cdot b + 1\cdot a,\ b)]$$
$$= [(a,b)]$$

(6) $[(-a,b)]$ is add. inverse $[(a,b)]$.

$$[(-a,b)] + [(a,b)]$$
$$= [((-a)\cdot b + b\cdot a,\ b^2)]$$
$$= [(0, b^2)]$$

$b \neq 0$ by assumption
$b^2 \neq 0$ as D is an ID

(10) $[(1,1)]$ is the multi. identity.

$$[(1,1)] = \{(a,a) : a \neq 0\}$$

$$[(1,1)] \cdot [(a,b)] = [(1\cdot a, 1\cdot b)]$$
$$= [(a,b)]$$

(11) $[(b,a)]$ is the multi. inverse of $[(a,b)] \neq 0_K$

$[(b,a)]$ is well-defined because

$[(a,b)] \neq 0_K \Rightarrow a \neq 0_D$ by ⊛

$$[(b,a)]\cdot[(a,b)] = [(ba, ab)] = 1_K$$

(13) Check $i: D \to K$

$$i(a) = [(a,1)]$$

is an injective homomorphism.

$i(a) = i(b) \quad a,b \in D$

$\Leftrightarrow [(a,1)] = [(b,1)]$

$\Leftrightarrow (a,1) \sim (b,1)$

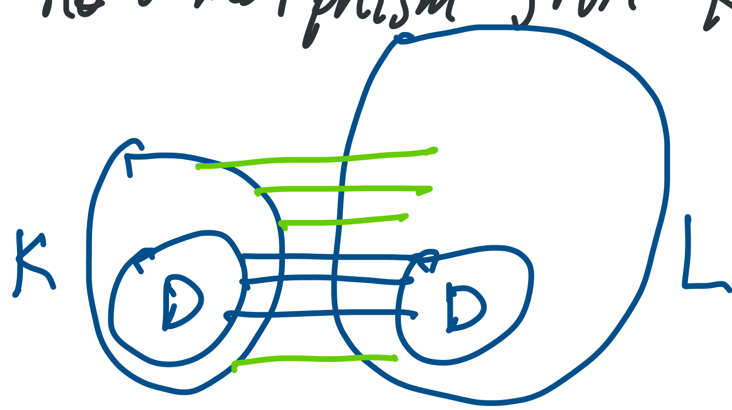
$\Leftrightarrow a\cdot 1 = 1\cdot b \Leftrightarrow a = b$

$$\begin{aligned} i(a+b) &= [(a+b, 1)] \\ &= [(a\cdot 1 + 1\cdot b, 1^2)] \\ &= [(a,1)] + [(b,1)] \\ &= i(a) + i(b) \end{aligned}$$

$$\begin{aligned} i(a\cdot b) &= [(a\cdot b, 1^2)] \\ &= [(a,1)][(b,1)] = i(a)\cdot i(b) \end{aligned}$$

Thm: Every ID D can be extended to a field K s.t. every element of K is a fraction of elements in D. A field with these properties ($D \subseteq K$ and every $x \in K$ is $\frac{a}{b}$ for $a, b \in D$) is a <u>field of fractions</u> of D.

Prop: Let D ($\mathbb{Z}$) be an ID. Suppose L ($\mathbb{C}$) is a field containing D, then there exists a unique injective homormorphism from K to L.



$\mathbb{C}$ is a field containing $\mathbb{Z}$

$\Rightarrow \mathbb{C}$ also contains $\mathbb{Q}$.

Prop: The field of fractions of D is unique up to isomorphism.