Polynomial Rings

Thursday, 10 March 2022 13:04

Polynomial functions vs polynomials

$f(x) = x^2$, $f: \mathbb{R} \to \mathbb{R}$

$(\mathbb{R}, \mathbb{R}, \{(a, a^2) : a \in \mathbb{R}\})$ "Look up table"

$= \{(0,0), (1,1), (2,4), \dots\}$

$$\left.\begin{array}{l}\text{Polynomial 1: } 0 \\ \text{Polynomial 2: } (x-0)(x-1)(x-2) \\ \qquad = x^3 - x\end{array}\right\} \mathbb{Z}_3[x]$$

Viewed as functions $\mathbb{Z}_3 \to \mathbb{Z}_3$ the two polynomials are equal.

For this course, we should consider them as <u>different</u> for having different coefficients.

Also, as functions, we need to respect the domain, so $f(i)$ is not valid.

x is an <u>indeterminate</u>,
i.e. a "formal symbol/element" in a (polynomial) ring, that we will describe how to manipulate it

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R is a ring (comm. w/ unity 1)

$R[x] = \{a_0 + a_1 x + \dots + a_n x^n : a_i \in R\}$

$\mathbb{Z}[x] = \{0, 2, 3-2x, x^3 - 3x + 1, \dots\}$

$(a_0 + a_1 x + \dots + a_n x^n)$ $\quad m \leq n$

$+ (b_0 + b_1 x + \dots + b_m x^m)$

$= (a_0 + b_0) + (a_1 + b_1) x + \dots$

$+ (a_m + b_m) x^m + a_{m+1} x^{m+1} + \dots + a_n x^n$

$(a_0 + a_1 x + \dots + a_n x^n)$

$\cdot (b_0 + b_1 x + \dots + b_m x^m)$

$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + \dots$

$+ (a_n b_{m-1} + a_{n-1} b_m) x^{m+n-1} + a_n b_m x^{m+n}$

Prop: If R is an ID (e.g. R is a field), then $R[x]$ is also an ID.

Proof: Suppose $(a_0+\dots+a_nx^n)$, $(b_0+\dots+b_mx^m)\neq 0$, $a_n, b_m \neq 0$.

Then $(a_0+\dots+a_nx^n)(b_0+\dots+b_mx^m)$

$$= a_0b_0+\dots+\underbrace{a_nb_m}_{\neq 0 \text{ b/c } R \text{ is an ID}}x^{n+m} \neq 0 \quad \square$$

Non-example: $R=\mathbb{Z}_4$, then $2x \cdot 2x = 0$ in $R[x]$

What we can't say:

"Plug $x=2$ into x^2+1"

"Solve $x^2-4=0$".

x is another element in $R[x]$, saying "$x=2$" is the same as saying "$3=2$"

Def: Let $\overset{\mathbb{R}}{F} \le \overset{\mathbb{C}}{E}$ be two fields, $\alpha \in E$. The <u>evaluation homomorphism</u>

$ev_\alpha : F[x] \to E$ is given by

$$ev_\alpha(\overbrace{a_0 + a_1 x + \dots + a_n x^n}^{f(x)})$$ Not a function

$$= a_0 + a_1\alpha + \dots + a_n\alpha^n$$

We <u>can</u> say:

"α is root/zero of $f(x)$"

if $f(x) \in \ker(ev_\alpha)$

Now we can say

i is a root of $x^2+1 \in \mathbb{R}[x]$

because $x^2+1 \in \ker(ev_i)$ where

$ev_i : \mathbb{R}[x] \to \mathbb{C}$ ← Free to change this field, avoiding the domain issue