Lecture 1.1 Binary Operations §2,3

A group is a set together with a binary operation satisfying three axioms TBC

Def 2.1 A binary operation $*$ on a set S is a function $* : S \times S \to S$.
(a, b)

Short hand notation: $*(a,b) =: a * b \in S$

Examples 1) $S = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$ and $* = +$ or $\times$ are binary operations on S

2) $S = \{f : \mathbb{R} \to \mathbb{R}\}$ and $* = +, \times, \circ$ are binary operations on S

function composition $f \circ g(x) = f(g(x))$

3) $S = Mat_{nm}(\mathbb{R})$ $* = +$ is a binary operation

- if $n = m$ then $* = \times$ is a binary operation.

! if $S = Mat(\mathbb{R})$ (matrices of any size) $* = +, \times$ are NOT binary operations

4) Non-example $S = \mathbb{N}$ and $* = \div$ not a binary operation on S.

$a * b := a/b \notin \mathbb{N}$ for $b \nmid a$ ("not dividing") or $b = 0$.

Def 2.4 Let $*$ be a binary operation on S and $H \subseteq S$

H is closed under $*$ if $a * b \in H$ $\forall a, b \in H$.

Ex 1) If $S = \mathbb{C}$ and $* = +$ then $H = \mathbb{N}, \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R} \subseteq S$ are closed under $*$. $H' = \{n \in \mathbb{N} \mid n \text{ odd}\} \subseteq \mathbb{C}$ is not closed under $+$.

2) $H = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous}\} \subseteq \{f : \mathbb{R} \to \mathbb{R}\} = S$ is closed under $* = +, \times, \circ$ ($\leftarrow$ composition)

Properties of binary operations

Def 2.11 A binary operation $*$ on S is commutative if

$$a * b = b * a \quad \forall\ a, b \in S.$$

Ex. 1) $S = Mat_n(\mathbb{R})$ with $* = \cdot$ is not commutative. 2) $S = \{f : \mathbb{R} \to \mathbb{R}\}$ then $\circ$ is not commutative.

Def 2.12 A binary operation $*$ on S is associative if

$$a * (b * c) = (a * b) * c \quad \forall\ a, b, c \in S$$

Ex 1) $a * b = a - b$ is not associative. Try it out!
$S = \mathbb{Z}$

Thm 2.13 Let $f, g, h : S \to S$ be functions then

$$f \circ (g \circ h) = (f \circ g) \circ h$$

ie. function composition is associative

Proof $\forall\ x \in S$ $\quad f \circ (g \circ h)(x) = f(g(h(x))) = (f \circ g) \circ h(x)$ $\square$

Keeping track of binary operations

Ex 2.14 Tables

1) $S = \mathbb{N}$ $\quad * = \times$

Multiplication table

×	1	2	3	4	......
1	1	2	3	4	
2	2	4	6	8	
3	3	6	9	12	
4	4	8	12	16	
⋮					

2) $S = \{a, b, c, d, e\}$

$*$ binary operation

2.26 Table

*	a	b	c	d	e
a	a	b	c	b	d
b	b	c	a	e	c
c	c	a	b	b	a
d	b	e	b	e	d
e	d	b	a	d	c

$a * b = b$

$b * e = c$

3) Is $*$ commutative?

2.17 Table

*	a	b	c	d
a	b	c	a	a
b	d	a	b	c
c	a	c	d	d
d	a	b	b	c

← Not symmetric along diagonal

$\Rightarrow *$ is not commutative

2.18 Table

*	a	b	c	d
a	b	d	a	a
b	d	a	c	b
c	a	c	d	b
d	a	b	b	c

$*$ is commutative if the mult. table is symmetric along the diagonal.

§ 3 Isomorphisms of binary structures

A binary structure is a pair $(S, *)$ with S a set and $*$ a binary operation.

Def 3.7 An isomorphism of binary alg. structures $(S, *)$ and $(S', *')$ is a one-to-one ("injective") and onto ("surjective") function $\phi : S \to S'$ such that

homomorphism property:

$$\phi(x * y) = \phi(x) *' \phi(y) \quad \forall x, y \in S$$

($x, y \in S$; $x * y \in S$; $\phi(x * y) \in S'$; $\phi(x), \phi(y) \in S'$; $\phi(x) *' \phi(y) \in S'$)

Write: $(S, *) \cong (S', *')$ when isomorphic.

Example 1)

$S = \{a, b, c\}$ $*$ $\quad \phi(a * b) = \phi(a)$

$\phi \downarrow \quad \downarrow \quad \downarrow$ $\quad = \#$

$S' = \{\#, \$, \&\}$ $*'$ $\quad = \# *' \$$

$S'' = \{x, y, z\}$ $*''$ $\quad = \phi(a) *' \phi(b)$

3.1 Table

*	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

3.2 Table

*'	#	$	&
#	&	#	$
$	#	$	&
&	$	&	#

3.3 Table

*''	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

Ex Find $\phi' : S' \to S''$ which is an isomorphism.

2) $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic

$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.

$\exp : \mathbb{R} \to \mathbb{R}^+$

$x \mapsto e^x$

exp is a bijection since its inverse function is ln (nat. log).

$\exp(x+y) = e^{x+y} = e^x \cdot e^y = \exp(x) \cdot \exp(y)$ $\forall x, y \in \mathbb{R}$.

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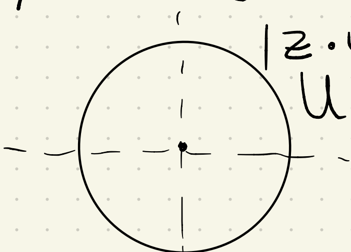
3) $U = \{z \in \mathbb{C} \mid |z| = 1\}$



$|z \cdot w| = |z| \cdot |w|$

U is closed under mult.

$(U, \cdot)$

$\mathbb{R}_{2\pi} = \{x \in \mathbb{R} \mid 0 \leq x < 2\pi\}$

$$x +_{2\pi} y := \begin{cases} x + y & \text{if } x+y < 2\pi \\ x + y - 2\pi & \text{if } x+y \geq 2\pi \end{cases}$$

Exercise Show $(U, \cdot) \cong (\mathbb{R}_{2\pi}, +_{2\pi})$