Algebraic Extensions §31

Recall E is a field extension of F if $F \leq E$

If $F \leq E$ are fields $\alpha \in E$ is algebraic over F if α is a zero of some $f(x) \in F[x]$

Ex - any $\alpha \in F \leq E$ is alg. over F $x - \alpha \in F[x]$

- $\mathbb{Q} \leq \mathbb{C}$ then $\sqrt{2}, i, \sqrt[k]{r}, \ldots$ algebraic / $\mathbb{Q}$
- If $\alpha \in E$ is algebraic $F \leq F(\alpha) \leq E$ and every $\beta \in F(\alpha)$ is algebraic over F.

$$F(\alpha) \cong \frac{F[x]}{\langle irr(\alpha, F) \rangle}$$

Def 31.1, 31.2 An extension E of F is algebraic if $\forall \alpha \in E$ is algebraic / F.

If an extension E of F is an n-dimensional vector space over F, say E is a finite ext'n of degree n over F. The degree of E over F is denoted $[E:F] = \dim_F(E)$

Ex $F(\alpha)$ is algebraic over F of degree $[F(\alpha):F] = \deg(\operatorname{irr}(\alpha, F))$.

Last time The field extension $F(\alpha)$ is a vector space over F:

Thm 30.23 Let $E \supseteq F$ and suppose $\alpha \in E$ is alg $/F$

If $\deg(\alpha, F) = n$ then $F(\alpha)$ is an n-dim'l vector space over F with basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$

Also every elt β of $F(\alpha)$ is algebraic $/F$

and $\deg(\beta, F) \leq \deg(\alpha, F)$.

Proof sketch 1) $\alpha = x + \langle \operatorname{irr}(\alpha, F) \rangle$

$$\begin{array}{ccc} F(\alpha) & \ni & b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1} \\ \downarrow & & \downarrow \\ F^n & \ni & (b_0, \ldots, b_{n-1}) \end{array}$$

2) $\left.1, \beta, \beta^2, \ldots, \beta^n\right]$ $n+1$ elts in an n-dim'l vector space must be linearly dependent.

$$c_0 + c_1\beta + c_2\beta^2 + \cdots + c_n\beta^n = 0$$

$c_i \in F$ not all zero. $\Rightarrow \beta$ is zero of $g(x) = c_0 + c_1x +$

Thm 31.3 If $[E:F] = n < \infty$ then E is alg. over F.

Proof Let $\alpha \in E$, then $1, \alpha, \alpha^2, \ldots, \alpha^n \in E$, $n+1$ elts in a vector space of n dimensions $\xRightarrow{\text{Thm 30.19}}$

$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0 \quad \text{for } a_i \in F \text{ not all zero.}$$

$\Rightarrow \alpha$ is a zero of $f(x) = a_n x^n + \cdots + a_0 \Rightarrow$ α is alg over F $\square$.

What about the converse? If E alg over F we may have $[E:F]$ not finite

Ex. $(((\mathbb{Q}(\sqrt{2}))(\sqrt{3}))(\sqrt{5}))(\sqrt{6}) \ldots \ldots$ ← field containing $\sqrt{n}$ $n \in \mathbb{Z}^+$ not finite

Example $\mathbb{Q}(\sqrt{2})$ is an extension of $\mathbb{Q}$

$\operatorname{irr}(\sqrt{2}, \mathbb{Q}) = x^2-2 \Rightarrow [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ and $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.

$$\mathbb{Q}(\sqrt{2}) = \{a_0 + a_1\sqrt{2} \mid a_0, a_1 \in \mathbb{Q}\}. \quad (*)$$

Notice $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ and $\operatorname{irr}(\sqrt{3}, \mathbb{Q}(\sqrt{2})) = x^2-3$.
so $\{1, \sqrt{3}\}$ is a basis for $(\mathbb{Q}(\sqrt{2}))(\sqrt{3})$ over $\mathbb{Q}(\sqrt{2})$

$$\begin{aligned}
(\mathbb{Q}(\sqrt{2}))(\sqrt{3}) &= \{b_0 + b_1\sqrt{3} \mid b_0, b_1 \in \mathbb{Q}(\sqrt{2})\} \\
&= \{(a_0 + a_1\sqrt{2}) + (a_0' + a_1'\sqrt{2})\sqrt{3} \mid \underbrace{a_0, a_1, a_0', a_1'}_{\text{unique}} \in \mathbb{Q}\}. \\
&= \{a_0 + a_1\sqrt{2} + a_0'\sqrt{3} + a_1'\sqrt{6} \mid a_0, a_1, a_0', a_1' \in \mathbb{Q}\}.
\end{aligned}$$

$\Rightarrow \{1, \sqrt{2}, \sqrt{3}, \sqrt{2}\cdot\sqrt{3}\}$ is a basis for $\mathbb{Q}(\sqrt{2})(\sqrt{3})$ over $\mathbb{Q}$. $[\mathbb{Q}(\sqrt{2})(\sqrt{3}) : \mathbb{Q}] = 4$.

$\mathbb{Q}(\sqrt{2}+\sqrt{3})$

$\parallel$

$$\begin{tikzcd}[column sep=small]
& \mathbb{Q}(\sqrt{2})(\sqrt{3}) = \mathbb{Q}(\sqrt{3})(\sqrt{2}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash] & & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash] \\
& \mathbb{Q} &
\end{tikzcd}$$

<u>Aside</u> $\beta = \sqrt{2}+\sqrt{3} \in \mathbb{Q}(\sqrt{2})(\sqrt{3})$.

$$\beta^2 = (\sqrt{2}+\sqrt{3})^2 = 2 + 2\sqrt{6} + 3$$

$$\beta^2 - 5 = 2\sqrt{6}$$

$$(\beta^2-5)^2 = 24 \Rightarrow \beta^4 - 10\beta^2 + 25 = 24$$

$$\Rightarrow \beta^4 - 10\beta^2 + 1 = 0 \Leftarrow \text{irreducible}$$

$\operatorname{irr}(\beta, \mathbb{Q}) = x^4 - 10x^2 + 1 \Rightarrow \deg(\beta, \mathbb{Q}) = 4$.

$\Rightarrow \mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2})(\sqrt{3})$. $\Leftarrow$ is also simple

Thm 31.4 Let $K \supseteq E \supseteq F$ be fields then with $[K:E]$ and $[E:F]$ finite then

$$[K:F] = [K:E][E:F]$$ in particular K is finite over F.

Proof Sketch Suppose $\{\alpha_1, \dots, \alpha_n\}$ is a basis for E over F and $\{\beta_1, \dots, \beta_m\}$ is a basis for K over E. Then

$$K = \{b_1\beta_1 + \dots + b_n\beta_n \mid b_i \in E\}.$$

$$= \{(a_{11}\alpha_1 + a_{21}\alpha_2 + \dots + a_{n1}\alpha_n)\beta_1 + \dots + (a_{1n}\alpha_1 + \dots + a_{nn}\alpha_n)\beta_n \mid a_{ij} \in F\}$$

$= \{ \sum a_{ij} \alpha_i \beta_j \mid a_{ij} \in F \}$. $\Rightarrow \{\alpha_i \beta_j\}$ span K over F

Also $\{\alpha_i \beta_j\}$ are independent over F $\Rightarrow$
$\{\alpha_i \beta_j\}$ is a basis for K over F

$\Rightarrow [K : F] = |\{\alpha_i \beta_j\}| = [K:E][K:F]$. $\square$.

Corollaries

31.6 If $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_r$ are finite field extensions
$[F_r : F_1] = [F_r : F_{r-1}][F_{r-1} : F_{r-2}] \ldots [F_2 : F_1]$.

31.7 If $E \supseteq F$ and $\alpha \in E$ is alg. over F then
$\forall \beta \in F(\alpha)$ deg(β, F) divides deg(α, F)

Proof 31.7 $F(\alpha) \ni \beta$ and $F \subseteq F(\beta) \subseteq F(\alpha)$.

$$\Rightarrow [F(\alpha) : F] = [F(\alpha) : F(\beta)][F(\beta) : F]$$

$$[F(\alpha):F] = \deg(\alpha, F), \quad [F(\beta):F] = \deg(\beta, F).$$

Ex 31.10 Consider $2^{1/3}, 2^{1/2} \in \mathbb{R}$

$\text{irr}(2^{1/3}, \mathbb{Q}) = x^3 - 2 \Rightarrow [\mathbb{Q}(2^{1/3}) : \mathbb{Q}] = 3$

$\text{irr}(2^{1/2}, \mathbb{Q}) = x^2 - 2 \Rightarrow [\mathbb{Q}(2^{1/2}) : \mathbb{Q}] = 2$

By Cor 31.7 $2^{1/2} \notin \mathbb{Q}(2^{1/3})$ and $\deg(2^{1/2}, \mathbb{Q}(2^{1/3})) = 2$.

So $[\mathbb{Q}(2^{1/3}, 2^{1/2}) : \mathbb{Q}] = 6$ and

$\{1, 2^{1/2}, 2^{1/3}, 2^{1/2+1/3}, 2^{2/3}, 2^{1/2+2/3}\}$ is a basis for $\mathbb{Q}(2^{1/3}, 2^{1/2})$ over $\mathbb{Q}$.

Notice that $2^{1/6} \in \mathbb{Q}(2^{1/3}, 2^{1/2})$

and $\deg(2^{1/6}, \mathbb{Q}) = 6$ $\quad$ $x^6 - 2$ is irreducible by Eisenstein.

$$\begin{tikzcd}
\mathbb{Q}(2^{1/3}, 2^{1/2}) \arrow[dd, dash, bend left=60, "6"] \arrow[d, dash] \\
\mathbb{Q}(2^{1/6}) \arrow[d, dash] \arrow[d, dash, bend left=60, "6"] \\
\mathbb{Q}
\end{tikzcd}$$

$$\mathbb{Q}(2^{1/3}, 2^{1/2}) = \mathbb{Q}(2^{1/6})$$

simple

Algebraic Closure

Let $E \supseteq F$. The algebraic closure of F in E is $\overline{F}_E = \{\alpha \in E \mid \alpha \text{ is alg. over } F\}$.

(This is a subfield of E by Thm 31.12).

Ex 1) If E is algebraic over F then $\overline{F}_E = E$

2) Recall $\alpha \in \mathbb{C}$ is algebraic number if algebraic over $\mathbb{Q}$. $\overline{\mathbb{Q}}_{\mathbb{C}}$ is the field of algebraic numbers

3) $E = \mathbb{Q}(x)$, then $\overline{\mathbb{Q}}_E = \mathbb{Q}$.

Def A field F is algebraically closed if every non-constant polynomial $f(x) \in F[x]$ has a zero in F.

Ex $\mathbb{C}$ is alg. closed (Thm 31.18)

Thm 31.15 A field F is alg closed if and only if every non-constant polynomial factors into linear factors.

Proof

An algebraically closed field has <u>no</u> algebraic extensions (Cor 31.16)

<u>Thm 31.17</u> Every field F has an algebraic closure $\overline{F}$. (Proof is difficult + omitted!)