Homomorphisms §13

G, G' groups

Def 13.1 A map $\phi : G \to G'$ is a **homomorphism** if $\phi(ab) = \phi(a)\phi(b) \ \forall a, b \in G$.

Ex Isomorphisms are homomorphisms where ϕ is a bijective

Ex. $\phi : G \to G'$ is the trivial homomorphism

$\forall g \mapsto e'$

identity in G'

$e' = \phi(ab) = \phi(a)\phi(b) = e'$

Prop Ex 13.2 If $\phi : G \to G'$ is a surjective homomorphism and G abelian, then G' is abelian

Proof let $a', b' \in G'$ then $\exists\ a, b \in G$ st. $\phi(a) = a'$, $\phi(b) = b'$.

$$\text{Then } a'b' = \phi(a)\phi(b) \overset{\text{homom.}}{=} \phi(ab) \overset{\text{abelian}}{=} \phi(ba) \overset{\text{homom.}}{=} \phi(b)\phi(a) = b'a'$$

So G' is abelian. $\square$

Examples

$n \geq 2$

13.3 $\phi : S_n \to \mathbb{Z}_2$, $+$. $\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ even} \\ 1 & \text{if } \sigma \text{ odd} \end{cases}$

Check: $\phi(\sigma_1 \sigma_2) = \phi(\sigma_1) + \phi(\sigma_2)$

13.4 Evaluation $F = \{ f : \mathbb{R} \to \mathbb{R} \}$ with addition let $c \in \mathbb{R}$

$\phi_c : (F, +) \to (\mathbb{R}, +)$. $\phi_c(f) = f(c) \in \mathbb{R}$.

$$\phi_c(f+g) = (f+g)(c) = f(c) + g(c) = \phi_c(f) + \phi_c(g)$$

Ex 13.6 $\phi : GL_n(\mathbb{R}) \to \mathbb{R}^\times$

$$A \mapsto \det(A).$$

$$\phi(AB) = \det(AB) = \det(A)\det(B) = \phi(A)\phi(B)$$

Ex 13.8 Projection homomorphism

$$G = \prod_{i=1}^{n} G_i \qquad \pi_i : G \to G_i$$

$$\pi_i((g_1, \ldots, g_n)) = g_i$$

$$\pi_i((g_1, \ldots, g_n)(g_1', \ldots, g_n')) = g_i g_i' = \pi_i(g_1, \ldots, g_n)\pi_i(g_1', \ldots, g_n')$$

13.10 "Reduction modulo n" $(\mathbb{Z}, +)$ $(\mathbb{Z}_n, +_n)$

$$\begin{aligned} r : \mathbb{Z} &\to \mathbb{Z}_n \\ m &\mapsto r \end{aligned}$$ where r is remainder when dividing by n.

<u>Exercise</u>: check homomorphism property (or see text).

<u>Exercise</u>: come up with more examples?

Properties of Homomorphisms

Def 13.11 Let $\phi : X \to Y$ be a map of sets.

For $A \subseteq X$ $B \subseteq Y$.

$\phi[A] := \{ \phi(a) \mid a \in A \} \subseteq Y$ image of A under ϕ.

$\phi[X]$ is range of ϕ.

$\phi^{-1}[B] := \{ x \in X \mid \phi(x) \in B \}$ inverse image of B.

Thm 13.12 Let $\phi : G \to G'$ be a group homomorphism

1. If $e \in G$ identity then $\phi(e) = e' \in G'$ identity

2. If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$

3. If $H \leq G$, then $\phi[H] \leq G'$

4. If K' is a subgroup in $G' \cap \phi[G]$ then $\phi^{-1}[K']$ is a subgroup of G.

Proof

1. If $e \in G$ identity then $\phi(e) = e' \in G'$ identity

$$\phi(a) = \phi(ae) = \phi(a)\phi(e) \Rightarrow \phi(e) = e' \in G'$$

by uniqueness of identity.

2. If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$

$$\phi(a^{-1})\phi(a) = \phi(aa^{-1}) = \phi(e) = e' \in G'.$$

$\Rightarrow [\phi(a)]^{-1} = \phi(a^{-1})$ by uniqueness of inverses.

3. If $H \leq G$, then $\phi[H] \leq G'$

closed: $a', b' \in \phi[H] \Rightarrow a' = \phi(a) \quad b' = \phi(b) \quad a, b \in G$

$\Rightarrow a'b' = \phi(a)\phi(b) = \phi(\underbrace{ab}_{\in H}) \in \phi[H]$.

identity: $e \in H$ and by 1 $\phi(e) = e' \in \phi[H]$.

inverses: $a' \in \phi[H] \Rightarrow a' = \phi(a) \quad a \in H + a^{-1} \in H$

$(a')^{-1} = \phi(a^{-1}) \in \phi[H]$.

4. see text book.