Homomorphism cont. § 13

Recall Def 13.1 A map $\phi : G \to G'$ between groups G, G' is a homomorphism if

$$\phi(ab) = \phi(a)\phi(b) \quad \forall \; a, b \in G.$$

Thm
Properties of homomorphisms If $\phi : G \to G'$ is a homomorphism

1) $\phi(e) = e'$ $\quad e, e'$ identities of G and G' resp.
2) $\phi(a)^{-1} = \phi(a^{-1}) \quad \forall \; a \in G$.
3) If $H \leq G$ then $\phi[H] \leq G'$
4) if $K' \leq \phi[G] \leq G'$ then $\phi^{-1}[K'] = \{ a \in G \mid \phi(a) \in K' \} \leq G$

Def 13.13 Let $\phi: G \to G'$ be a homomorphism of groups

Then the subgroup of G

trivial subgp of G'

$$\underline{\mathrm{Ker}(\phi)} := \phi^{-1}[\{e'\}] = \{a \in G \mid \phi(a) = e'\} \leq G.$$

is the kernel of ϕ

exercise: Prove directly that $\mathrm{Ker}\,\phi$ is a subgroup. 1) closed 2) contains identity 3) contains inverses

Alternatively apply pt 4) from homom. prop. using $K' = \{e'\}$

Ex 1) Let $\phi: \mathbb{Z}^{,+} \to \mathbb{Z}_n^{,+} = \{0, \ldots, n-1\}$. $\phi(m) = r$ where $m = qn + r$.

$\mathrm{Ker}\,\phi = \{kn \mid k \in \mathbb{Z}\} = n\mathbb{Z}$.

2) Let $\phi: S_n \to \underset{\{0,1\}}{\underset{\parallel}{\mathbb{Z}_2}}$ $\quad \phi(\sigma) = \begin{cases} 0 & \sigma \text{ even} \\ 1 & \sigma \text{ odd} \end{cases}$ $\quad \mathrm{Ker}\,\phi = \{\text{even perms}\} = A_n \leq S_n$

Left and Right cosets of $\ker \phi$.

Thm 13.15 Let $\phi : G \to G'$ be a group homomorphism and set $H = \ker \phi$. Then

$$aH = Ha = \phi^{-1}[\phi(a)] = \{x \in G \mid \phi(x) = \phi(a)\}.$$

Remark: This was true in the previous two examples (we knew that for other reasons).

Proof To show sets $A = B$ we can show $A \subseteq B$ & $B \subseteq A$.

See book!!

$$\underline{\phi^{-1}[\phi(a)] = a\overset{\overset{\ker\phi}{\shortparallel}}{H}}$$

$x \in \phi^{-1}[\phi(a)] \iff \phi(x) = \phi(a)$

$\iff \phi(a)^{-1}\phi(x) = e' \overset{\text{homo. prop.}}{\iff} \phi(a^{-1}x) = e' \iff a^{-1}x \in H.$

$\iff x \in aH$

$$\underline{\phi^{-1}[\phi(a)] = Ha}$$

$x \in \phi^{-1}[\phi(a)] \iff \phi(x) = \phi(a)$

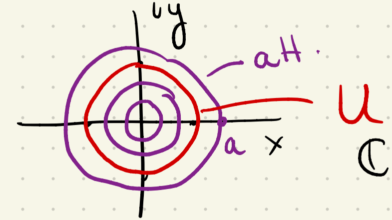
$\phi(x)\phi(a)^{-1} = e' \underset{\text{prop.}}{\overset{\text{homo}}{\iff}} \phi(xa^{-1}) = e' \iff xa^{-1} \in H$

$\iff x \in Ha.$ We proved $aH = \phi^{-1}[\phi(a)] = Ha$

$\square$

Examples 1) $\phi : \mathbb{C}^* \longrightarrow \mathbb{R}^*$ $\quad \phi(z) = |z|$ is a homomorphism (to check)

$$\mathrm{Ker}(\phi) = \{ z \in \mathbb{C}^* \mid \phi(z) = 1 \}$$
$$|z| = 1$$
$$= U$$



2) $\phi : GL_n(\mathbb{R}) \to \mathbb{R}^*$ $\quad A \mapsto \det A$ is a homomorphism

$\mathrm{Ker}\,\phi = \{ A \mid \det A = 1 \} = SL_n(\mathbb{R}) = H$ ← $A \in H$ linear trans. preserving area & orientation.

cosets $\quad AH = \{ B \mid \det B = \det A \}$.

e' is function $f(x) = 0 \;\; \forall x \in \mathbb{R}$.

3) $\phi : D \to F \quad D = \{ f : \mathbb{R} \to \mathbb{R} \text{ differentiable} \} \quad F = \{ f : \mathbb{R} \to \mathbb{R} \}$

$\phi(f) = f' \qquad D, F$ are groups with addition.

$\phi(f+g) = [f+g]' = f' + g' = \phi(f) + \phi(g)$. $\quad \mathrm{Ker}\,\phi =$ constant functions.

4) $G=(\mathbb{R}^m,+)$ $\quad G'=(\mathbb{R}^n,+)$. $\quad A$ $\quad n\times m$ matrix.

$$\phi_A : \mathbb{R}^m \to \mathbb{R}^n \qquad {}_n\Big(\overset{m}{\qquad\qquad}\Big)\begin{pmatrix} \vdots \end{pmatrix}_m = \Big(\quad\Big)_n .$$

$$\phi_A\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$\operatorname{Ker}\phi_A = \operatorname{Null} A.$$

Def 13.19 A subgroup H of a group G is <u>normal</u> if its left and right cosets are equal.

ie. $\forall\ a \in G \quad aH = Ha$

Ex. 1) If G is abelian, every $H \leq G$ is <u>normal</u>

2) If $[G:H] = \#$ of left cosets of H $= 2$ then $H \leq G$ normal.

Corollary 13.20 If $\phi : G \to G'$ is a homomorphism, then $\operatorname{Ker}\phi \leq G$ is a normal subgroup.

Corollary 13.18 A group homomorphism $\phi: G \to G'$ is one to one (injective) if and only if $\operatorname{Ker}(\phi) = \{e\}$

Proof If $\operatorname{Ker}(\phi) = e$, let $H = \operatorname{Ker}\phi$ then

$$aH = \{x \in G \mid \phi(x) = \phi(a)\} = a\{e\} = a$$

so ϕ is one to one.

If ϕ is one to one by definition $\operatorname{Ker}\phi = \{e\}$.

Ex 1) $\phi: \mathbb{R}^* \to \mathbb{C}^*$ is a homomorphism +
$x \mapsto x$ is injective.

Quiz 3) $H \leq G \quad x, a \in G$

Then $x \in aH \iff xH = aH = \{ah \mid h \in H\}$

$x \cdot e \in xH = aH \Longleftarrow xH = aH$

$\Rightarrow x \in aH$

$x \in aH$ we know $x \in xH$ so

$x \in aH \cap xH \neq \emptyset$ but distinct cosets are disjoint

$\Rightarrow aH = xH$. When we can write a coset as aH we say a is a representative of the coset.

Homomorphisms & Isomorphisms.

Prop A map $\phi: G \to G'$ is an isomorphism of groups if:

1) ϕ is a homomorphism
2) $\text{Ker}(\phi) = e$
3) ϕ is onto.

Factor Groups "Quotient groups" § 14

Recall A group G is a set with binary operation
* satisfying axioms G_1, G_2 & G_3.

Goal Define a group on the set of cosets of H in G.

$$G/H := \{H, a_1H, a_2H, \ldots\ldots\}$$

⚠ The tricky thing will be figuring When + How we can define an operation on the cosets.

let $G/H := \{ H, a_1 H, a_2 H, \dots \}$

Define (?) $aH * bH := abH$

Does this make sense? Recall $a'H = aH \Leftrightarrow a^{-1}a' \in H$

$\Leftrightarrow a' \in aH$

For $*$ above to be well defined we need

$$(aH) * (bH) = abH$$

$$\shortparallel \qquad\quad \shortparallel$$

$$(a'H) * (b'H) = a'b'H$$

may not be equal!

Example $G = S_3$ $|G| = 6$ and $H = \langle (1,2) \rangle$. $|H| = 2$ $\rho_1 = (1,2,3)$.

$$eH = (1,2)H \quad \rho_1 H = \tau_1 H \quad \rho_2 H = \tau_2 H$$

$$G/H = \{ \{e, (1,2)\}, \{\underset{\rho_1}{(1,2,3)}, \underset{\tau_1}{(1,3)}\}, \{\underset{\rho_2}{(1,3,2)}, \underset{\tau_2}{(2,3)}\} \}.$$

using the suggested rule $aH * bH = abH$

$$(eH) * (\rho_1 H) = e\rho_1 H = \rho_1 H$$

$$\Vert$$

$$((1,2)H) * (\rho_1 H) = (1,2)(1,2,3)H = \cancel{(1)}(2,3)H$$

Therefore $*$ is not well-defined

H is not a normal subgp & this why $*$ doesn't work.

Thm 14.4 Let H be a subgroup of a group G. Then left coset multiplication is well defined by

$$(aH)(bH) = abH$$

if and only if H is a normal subgroup.

Proof 1) $aH\,bH = abH$ is well defined $\Rightarrow$ H normal.

Well defined: $abH = a'b'H$ whenever $aH = a'H$ & $bH = b'H$.

Normal: $aH = Ha \;\forall\, a \in G$ (ie. $aH \subseteq Ha$ & $Ha \subseteq aH$

$\underline{aH \subseteq Ha:}$

$$x \in aH \iff xH \overset{*}{=} aH \Rightarrow a^{-1}xH = H$$

$$\Rightarrow H = (aa^{-1})H \underset{\text{well defined}}{\overset{\text{coset mult is}}{=}} aHa^{-1}H \overset{*}{=} xHa^{-1}H \underset{\text{well defined}}{\overset{\text{coset mult}}{=}} xa^{-1}H$$

$$H = xa^{-1}H \iff xa^{-1} \in H \iff x \in Ha.$$

$\underline{Ha \subseteq aH:}$

$$x \in Ha \iff xa^{-1} \in H \iff xa^{-1}H = H \iff a^{-1}H \overset{*}{=} x^{-1}H$$

$$H = x^{-1}xH = x^{-1}HxH = a^{-1}HxH = a^{-1}xH$$

$$\Rightarrow a^{-1}xH = H \Rightarrow xH = aH \Rightarrow x \in aH$$

$aH = Ha$ when coset mult. is well defined

2): H normal $\Rightarrow$ $(aH)(bH) = abH$ is well defined

$aH\,bH = abH$ now take a', b' s.t.

(**) $a'H = aH$ and $b'H = bH$.

$a'H\,b'H = a'b'H$ we need to show that

$abH = a'b'H$.

Since (**) we know $a' \in aH \Rightarrow a' = ah_1$ $h_1, h_2 \in H$

$\Rightarrow b' = bh_2$

$a'b'H = (a(h_1 b)h_2)H$ now $h_1 b \in Hb = bH$ since

H is normal $\Rightarrow h_1 b = bh_3$ $h_3 \in H$.

$a'b'H = ab(h_3 h_2 H) = abH$ The product is well defined!

Corollary 14.5 Let H be a normal subgroup. Then the set of cosets of H form a group G/H under the operation

$$aH\,bH = abH.$$

Proof G1 $aH[bHcH] = abcH = [aHbH]cH$

G2 identity is $H = eH$ since $aHeH = aeH = aH$

G3 inverse of aH is $a^{-1}H$ since:

$$aHa^{-1}H = aa^{-1}H = eH = H.$$

Def 14.6 The group G/H is the factor group or quotient group of G by H.

Example + Thm 14.1

Let $\phi: G \to G'$ be a group homomorphism $\operatorname{Ker}\phi = H$. The cosets of H form a quotient group G/H. Also the map $\mu : G/H \to \phi[G]$ $\mu(aH) = \phi(a)$ is an isomorphism