Quotient groups: computations + examples

Def 14.6 Let H be a normal subgroup of G.

The group $G/H = \{H, aH, bH, \dots\}$ with operation $aH\,bH = abH$ is the quotient or factor group of G by H.

Recall H is a normal subgroup of G if $H \leq G$ and $gH = Hg \quad \forall\, g \in H$

Thm 14.13 (Equivalent conditions to be normal)

The following are equivalent conditions for a subgroup H to be a normal subgroup of G.

1) $ghg^{-1} \in H \quad \forall\ g \in G \quad h \in H$.

2) $gHg^{-1} = H$ for all $g \in G$

3) $gH = Hg$ for $\forall\ g \in G$.

1,2 $\Rightarrow$ A Normal subgroup H is sent to itself by the map $i_g : G \to G$ $\quad i_g(h) = ghg^{-1} \quad i_g[H] = H$ return to this in §36, 37

Example. The trivial subgroup $H = \{e\} \leq G$ is always normal

$G/\{e\} = \{a\{e\} \ \forall\ a \in G\} \xrightarrow{\phi} G \quad \phi(\{a\}) = a \in G$.

$a\{e\} = \{a\}$.

ϕ is an isomorphism.

Example modular arithmetic Ex. 14.2, 14.7

Consider: $H = n\mathbb{Z} \leq \mathbb{Z} = G$. $H = n\mathbb{Z}$ has cosets

$n\mathbb{Z} = \{kn \mid k \in \mathbb{Z}\}$.

$1 + n\mathbb{Z} = \{1 + kn \mid k \in \mathbb{Z}\} = (n+1) + n\mathbb{Z}$

$\vdots$

$(n-1) + n\mathbb{Z} = \{(n-1) + kn \mid k \in \mathbb{Z}\}$.

$\mathbb{Z}$ is abelian so all subgroups are normal. So $\mathbb{Z}/n\mathbb{Z}$ is a group.

Coset representation: $(r + kn) + n\mathbb{Z} = r + n\mathbb{Z}$

Group operation: $\underline{n = 5}$.

$(4 + 5\mathbb{Z}) + (3 + 5\mathbb{Z}) = (7 + 5\mathbb{Z}) = 2 + 1 \cdot 5 + 5\mathbb{Z} = 2 + 5\mathbb{Z}$.

instead of writing $a + n\mathbb{Z} = b + n\mathbb{Z}$ use $a \equiv b \mod n$.

$4 + 3 \equiv 7 \equiv 2 \mod 5$.

The map $\phi : \mathbb{Z} \to \mathbb{Z}_n = \{0, \dots, n-1\}$ $(\mathbb{Z}_n, +_n)$

$\phi(m) = r$ where $m = qn + r$ $0 \le r \le n-1$.

is a surjective homomorphism. $\operatorname{Ker} \phi = n\mathbb{Z}$

$$\mu : \mathbb{Z}/_{n\mathbb{Z}} \longrightarrow \mathbb{Z}_n$$

$$\underset{\Downarrow}{\mu(a + n\mathbb{Z})} = \phi(a)$$

$$\begin{array}{ccc} a + n\mathbb{Z} & & \phi(a) \\ \| & & \| \\ b + n\mathbb{Z} & & \phi(b) \end{array}$$

$$\operatorname{Ker} \mu = \{n\mathbb{Z} = H\} \le \mathbb{Z}/_{n\mathbb{Z}}. \quad \{\hat{e}\} = n\mathbb{Z}$$

$\leftarrow$ this group has identity $\hat{e} = n\mathbb{Z}$

$\Rightarrow \mu$ is injective $\Rightarrow \mu$ is an isomorphism and $\mathbb{Z}/_{n\mathbb{Z}} \cong \mathbb{Z}_n$

There is also a homomorphism $\gamma : \mathbb{Z} \to \mathbb{Z}/_{n\mathbb{Z}}$ with $\operatorname{Ker} \gamma = n\mathbb{Z}$

$\gamma(a) = a + n\mathbb{Z}$

Thm 14.9 Let $H \leq G$ be a normal subgroup of G. Then $\gamma : G \to G/H$ given by $\gamma(x) = xH$ is a homomorphism with kernel H.

Thm 14.11 Let $\phi : G \to G'$ be a group homomorphism with kernel H. Then $\mu : G/H \to \phi[G]$ given by $\mu(gH) = \phi(g) \in G'$ is an isomorphism. If $\gamma : G \to G/H$ is as above then $\phi(g) = \mu\gamma(g)$

In other words, $\exists$ a diagram of homomorphism $\Longrightarrow$

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\gamma"'] & & \phi[G] \\ & G/H \arrow[ur, "\mu"'] & \end{tikzcd}$$

which commutes, i.e. every homomorphism "factors" through the quotient group.

We'll return to conjugation, inner automorphisms §36, 37.

Recall Thm 11.12 Finitely generated abelian groups are direct products

$$\mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$$

p_i's not necess distinct primes

Prop Quotient groups of finitely generated abelian groups are finitely generated abelian groups.

Example 15.7 Compute $\underset{G}{(\mathbb{Z}_4 \times \mathbb{Z}_6)} / \underset{H}{\langle(0,1)\rangle} = \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$ $\quad |G| = 24$

$G/H = \{$ cosets of H in $G\}$. $\quad H = \{(0,0), (0,1), (0,2), \ldots (0,5)\}$ $|H| = 6$

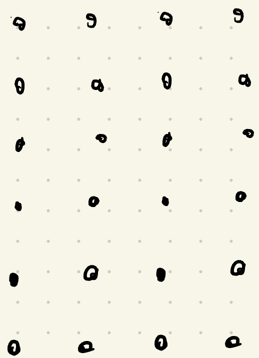
$|G/H| = [G:H] = \frac{24}{6} = 4$ $\quad (1,0)+H = \{(1,0), (1,1), \ldots\}$. $G/H = \langle (1,0)+H \rangle$

$G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. $\quad (2,0)+H = \{(2,0), \quad\}$ $\quad G/H \cong \mathbb{Z}_4$.

$(3,0)+H = \{(3,0), \ldots\}$

Example 15.10

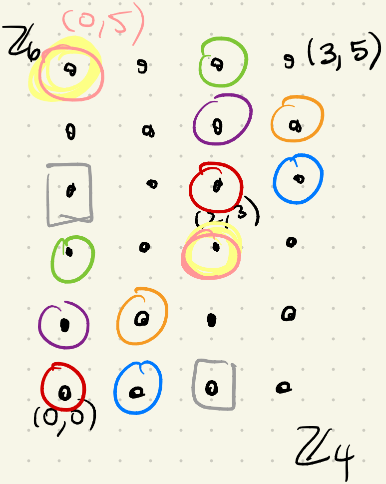
Compute $\mathbb{Z}_4 \times \mathbb{Z}_6 / (\langle 0,2 \rangle)$



Ex. 15.11

$\mathbb{Z}_4 \times \mathbb{Z}_6 / \langle (2,3) \rangle$, $\langle (2,3) \rangle = H$ $\qquad |G| = 24. \qquad |G/H| = 12.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$



$H = \{ (0,0), (2,3), \cancel{(4,6)} \}$ $\qquad |H| = 2. \qquad 2 \cdot 2 \cdot 3$

G/H is either $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_4 \times \mathbb{Z}_3$.

$(1,0) + H = \{ (1,0), (3,3) \}$ $\qquad (1,0) + [(1,0) + H] = \{ (2,0), (0,3) \}$

$(0,1) + H = \{ (0,1), (2,4) \}$

$(2,2) + H = \{ (2,2), (0,5) \}$.

$\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$

Chapter 11.

$(1,0) + H$ has order 4 in G/H. so

$[(1,0) + H] + [(1,0) + H] = (2,0) + H \neq H$

$[(1,0) + H] + [(1,0) + H] + [(1,0) + H] = (3,0) + H \neq H.$

$[(1,0) + H] + [(1,0) + H] + [(1,0) + H] + [(1,0) + H] = (0,0) + H = H.$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ has no elt of order 4.

$\Rightarrow G/H \cong \mathbb{Z}_4 \times \mathbb{Z}_3.$

Quiz Questions

In each case, compute G/H:

1) $G = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\quad |G| = 6$ $\quad H = \langle (1,0) \rangle$ $\quad |H| = 2$

$G/H \cong \mathbb{Z}_3$

$G/H = \{H, (0,1)+H, (0,2)+H\}$

2) $G = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\quad |G| = 6$ $\quad H = \langle (0,1) \rangle$

$|H| = 3 \quad |G/H| = 2 \quad G/H \cong \mathbb{Z}_2$.

3) $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\quad |G| = 9$ $\quad H = \langle (1,1) \rangle = \{(0,0), (1,1), (2,2)\}$

$G/H \cong \mathbb{Z}_3$

$= \{H, (1,0)+H, (0,1)+H\}$.

4) $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\quad H = \langle (1,0) \rangle$

$|G| = 9 \quad |H| = 3 \quad |G/H| = 3 \quad G/H \cong \mathbb{Z}_3$

Exercise 14.15 Compute: $(\mathbb{Z}_6 \times \mathbb{Z}_8)/\langle(4,4)\rangle$.

$|G| = 48$ $H = \{(0,0), (4,4), (2,0), (0,4), (4,0), (2,4)\}$

$|H| = 6$.

$|G/H| = 8$ G/H is either $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong$, $\mathbb{Z}_2 \times \mathbb{Z}_4 \not\cong$, $\mathbb{Z}_8$

(see Chapter 11 again).

The elt $(0,1) + H$ has order 4.

$$4(0,1) + H = (0,4) + H = H$$

No elt $(a,b) + H$ has order 8.

since $4(a,b) + H = H$ for any $(a,b) \in \mathbb{Z}_6 \times \mathbb{Z}_8$.

Hence we are not $\mathbb{Z}_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\Rightarrow G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_4.$$

Thm 15.8 let $G = H \times K$ direct product of groups G + K. Then $\overline{H} = \{(h,e) \mid h \in H\}$ is a normal subgroup and $G/\overline{H} \cong K$.

Proof $\pi_2 : H \times K \to K$ $\quad \ker \pi_2 = \overline{H} \Rightarrow \overline{H}$ is normal

π_2 is surjective by Thm 14.11 we have $G/\overline{H} \cong K$.

Thm 15.9 The quotient of a cyclic group is cyclic

Proof if $G = \langle a \rangle$ prove that $G/N = \langle aN \rangle$.

Simple Groups

Def 15.14 A group is simple if it is non-trivial and has no proper non trivial normal subgroups

Thm 15.15 The alternating group A_n is simple for $n \geq 5$.

Proof outlined in exercises

Classification of finite simple groups was a HUGE task completed over many years by over 100 authors!

Thm 15.16 Let $\phi : G \to G'$ be a homomorphism. If $N \leq G$ is normal then $\phi[N]$ is normal in $\phi[G]$. Also if $N' \leq \phi[G']$ is normal then $\phi^{-1}[N] \leq G$ is normal.

Proof Exercises 35, 36. (these are on the list)

Def 15.17 A maximal normal subgroup of G is a normal subgroup $M \neq G$ s.t. there is no N normal subgroup of G s.t. $M \neq N \neq G$

Thm 15.18 M is a maximal normal subgroup if and only if G/M is simple.

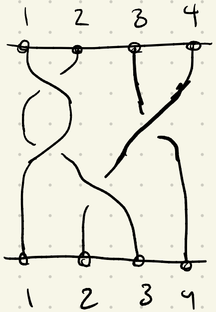
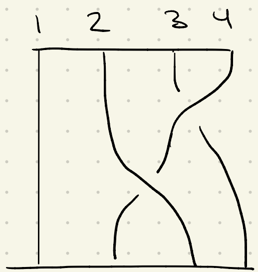
Proof Let M be a max normal subgroup. Suppose G/M is not simple i.e. $N' \leq G/M$ is normal. Then $M \lneq \gamma^{-1}[N'] \lneq G$ is normal in G where $\gamma : G \to G/M$ canonical homomorphism. contradiction.

Conversely, if $N \leq G$ is normal then $\gamma(N) \lneq G/M$ is normal by Thm 15.16 contradicting G/M simple $\square$.

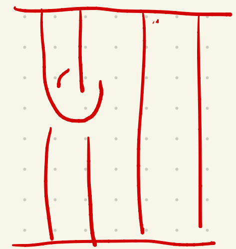
Extra topic

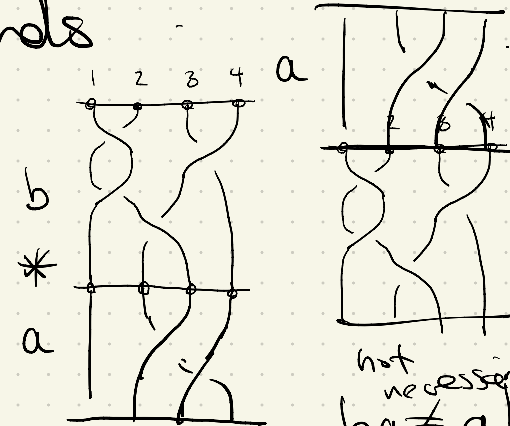
The braid group B_n on n "strands".

$n=4$

b  = 

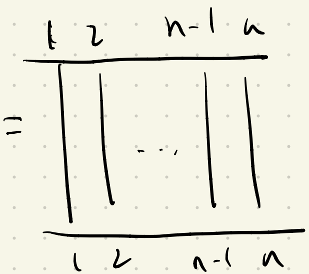
NOT A BRAID

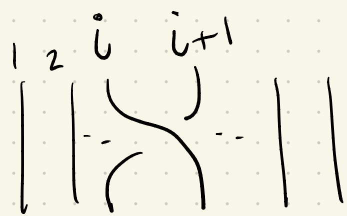




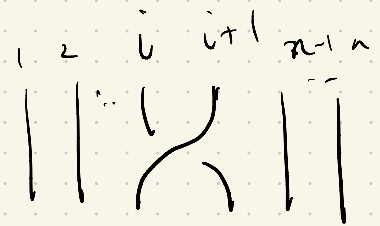
not necessarily $ba \neq ab$.

$*$ is associative.

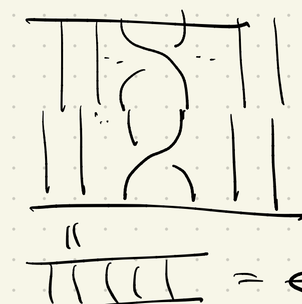
$e =$ 



σ_i



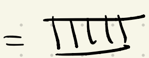
σ_i^{-1}



$= e$

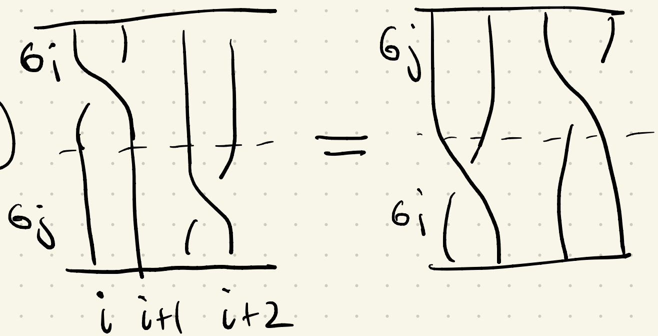
$\underline{\sigma_i^k \neq e}$  B_n is $\underline{\underline{\text{infinite}}}$

B_n is $\underline{\text{a group}}$.

- $*$ associative
- $e =$ 
- a^{-1} is a "flipped"

A braid B_n is a concatenation of "twists" σ_i, σ_i^{-1} for $i = 1, \dots, n-1$. There are braid relations

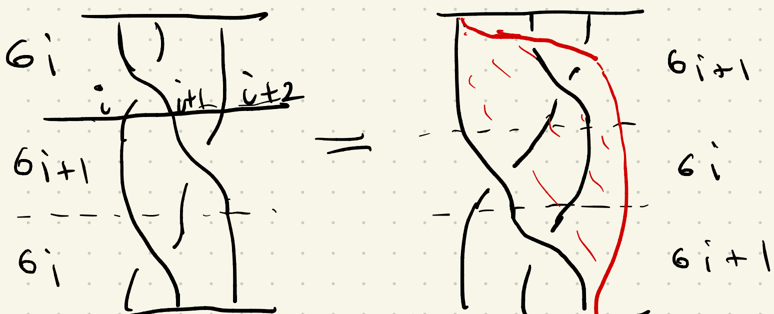
1



$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

$$|i-j| \geq 2$$

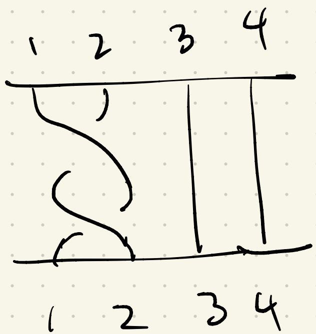
2



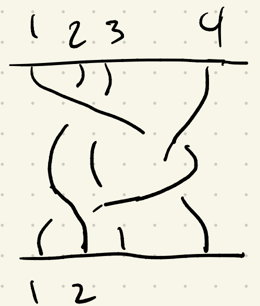
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

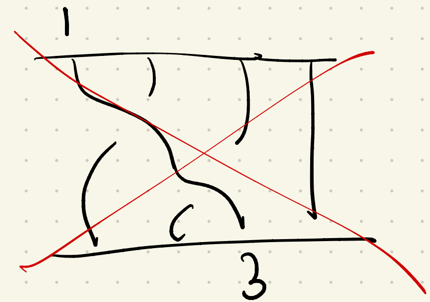
$\forall \quad i = 1, \dots, n-2$

There is a subgroup $P_n \leq B_n$ of "pure braids" $P_n = \{ b \mid b \text{ s.t. strand } i \text{ lands in position } i \}$



or





__Question__ What are the cosets of P_n?

Is P_n normal?

If so what is B_n / P_n?