

Group actions § 16

Def 16.1 Let X be a set and G be a group.

An action of G on X is a map $* : G \times X \rightarrow X$ s.t.

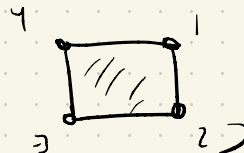
1. $e * x = x \quad \forall x \in X$

2. $(g_1 g_2) * x = g_1 * (g_2 * x) \quad \forall x \in X \text{ and } g_1, g_2 \in G$

Say X is a G -set.

Example. If X is any set then S_X acts on X

$$X = \{1, \dots, n\} \quad * : S_n \times X \rightarrow X$$
$$(g, i) \mapsto g(i).$$

- Same $H \leq S_X$ X is still an H -set.
- $X = G$ is a G set $*: G \times G \rightarrow G$ $(g_1, g_2) \mapsto g_1 g_2$
- D_4 acts on the square  also on $\{1, 2, 3, 4\}$,
see example 16.8 for another D_4 set X .
Geir Ellingsrud's note on D_4 acting on the square.
- The Rubik's cube group acts on the Rubik's cube.
- Any group G acts trivially on any set X .
 $*: G \times X \rightarrow X$ $gx = x \quad \forall g \in G \quad x \in X$.

Thm 16.3 Let X be a G -set. For all $g \in G$, the function $\phi_g: X \rightarrow X$ defined by $\phi_g(x) = gx$ for $x \in X$ is a permutation of X (ie $\phi_g \in S_X$)

Also $\phi: G \rightarrow S_X$ is a homomorphism satisfying

$$\phi(g) \cdot x = gx \quad \forall g \in G \quad x \in X$$

Proof See text.

- 1) Must show ϕ_g is a bijection (see Cayley's Thm)
- 2) Show ϕ satisfies homomorphism property.

Question What is the kernel of ϕ ?

$$\text{Ker } \phi = \{ g \in G \text{ st. } gx = x \quad \forall x \in X \}$$

Def • G acts faithfully on X if $\text{Ker } \phi = e$ where

$$\phi: G \rightarrow S_X$$

• A group G acts transitively on a G -set if $\forall x_1, x_2 \in X \exists g \in G \text{ st. } gx_1 = x_2$.

Corollary If G acts on X then $G/\text{Ker } \phi = H$ acts faithfully on X via $(aH)x = ax \quad \forall aH \in G/H$.

Task during the break

Find one example of a group action

we haven't see

Isotropy Subgroups

Def: let X be a G -set. Define

$$X_g = \{x \in X \mid gx = x\} \quad \text{and} \quad G_x = \{g \in G \mid gx = x\}.$$

"elements fixed by $g \in G$ "

"elements of G fixing x "

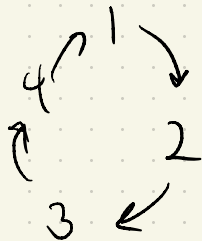
Thm 16.12 G_x is a subgroup of G

Proof 1) closed: $g_1, g_2 \in G_x$ then $(g_1 g_2)x = g_1(g_2 x) = g_1 x = x$
 $\Rightarrow g_1 g_2 \in G_x$

2) $ex = x \Rightarrow e \in G_x$ 3) If $g \in G_x$ then $x = ex = g^{-1} g x = g^{-1} x$
 $g^{-1} \in G_x$ □

Orbits Recall $G \in S_n$ we considered its cycles.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$$



equivalence
classes
"orbits"

Can think of $G = \langle (67) \rangle$ acting on $X = \{1, \dots, 6\}$.
we can generalize this to any group.

Thm 16.14 Let X be a G -set. For $x_1, x_2 \in X$

define $x_1 \sim x_2 \iff \exists g \in G$ s.t. $gx_1 = x_2$.

This is an equivalence relation on X .

Def Let X be a G -set. A cell in the partition of the equivalence relation is an orbit in X under G .

$$O_x = \{x' \in X \mid g \cdot x = x' \quad g \in G\}$$

Book writes $Gx \triangleq$

Example. If G acts transitively then $x_1 \sim x_2 \quad \forall x_1, x_2 \in X$.
 Transitive $\iff Gx = X$ for any x in X .

• Let $G \leq G'$ define an action of G on G' by
 $G \times G' \rightarrow G'$
 $(g, g') \mapsto gg'$
 What are the orbits?
 left cosets of G

Thm 16.16 let X be a G -set. Then

$$|\mathcal{O}_X| = (G : G_x) \leftarrow \# \text{ of cosets of } G_x \text{ in } G.$$

In particular, if $|G| < \infty$ then $|\mathcal{O}_X|$ divides $|G|$.

Proof.

$$\psi : \underbrace{\{G_x, g_1 G_x, g_2 G_x, \dots\}}_{\text{set of cosets}} \rightarrow \mathcal{O}_X$$

Proceed with caution:

suppose $g' G_x = g G_x$ then $g' = g \hat{g} \quad \hat{g} \in G_x$

$$\psi(g' G_x) = g \hat{g} x = g x = \psi(g G_x) \quad \text{well defined on ~~sets~~ cosets}$$

$$g G_x \mapsto g x$$

ψ is one to one:

$$gx = \psi(gG_x) = \psi(g'G_x) = g'x$$

then $g^{-1}g'x = x \Rightarrow g^{-1}g' \in G_x \Rightarrow g' \in gG_x$
 $\Rightarrow g'G_x = gG_x$

ψ is onto:

if $x_1 \in \mathcal{O}_x \exists g \in G$ s.t. $gx = x_1$ therefore

$$x_1 = \psi(gG_x)$$

□