UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2200 — Groups, Rings and Fields

Day of examination: PRACTICE EXAM

Examination hours: 4 hours –

This problem set consists of 2 pages.

Appendices: none

Permitted aids: all

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Justification must be provided for all solutions. Solutions can be submitted in English or Norwegian. The format may be in Latex or as scanned handwritten notes.

Problem 1 Group theory

1. (5 points) Let G be a cyclic group of order 12. List all of the subgroups of G.
2. (10 points) Consider the subgroup H of S_4 generated by the 4-cycle $(1, 2, 3, 4)$. Determine the number of cosets of H in S_4 and write down two of them as subsets of S_4. Is H a normal subgroup of S_4?
3. (10 points) Let p be a prime number and let N be a normal p-subgroup of a finite group G. Prove that $N \leq P$ for every Sylow p-subgroup P of G.

Problem 2 Ring theory

1. (5 points) Let $f(x) = x^2 - 1$. Show that the ring $R = \mathbb{Q}[x]/\langle f(x)\rangle$ is not an integral domain. Is R a field?
2. (10 points) Let R and S be commutative rings with unity and let $\Phi : R \to S$ be a ring homomorphism. Show that if Φ is surjective then the preimage of a maximal ideal is maximal. Give an example of rings R and S, a non-trivial ring homomorphism $\Phi : R \to S$, and an ideal $I \subset R$ such that I is maximal in R but $\Phi(I)$ is not maximal in S.
3. (10 points) Let R be a commutative ring with unit element 1. If J_1 and J_2 are two ideals, define $J_1 + J_2 = \{a_1 + a_2 \in R \mid a_1 \in J_1 \text{ and } a_2 \in J_2\}$. Show that $J_1 + J_2$ is an ideal of R. Find the kernel of the ring homomorphism $\Phi : R \to R/J_1 \times R/J_2$ defined by $\Phi(a) = (a + J_1, a + J_2)$.

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Problem 3 Finite fields

Consider $F = \mathbb{Z}_5$.

1. (5 points) Show the polynomial $f(x) = x^3 + x + 1 \in F[x]$ is irreducible over F.

2. (10 points) Explain why $f(x)$ divides the polynomial $x^{5^3} - x$ over F.

3. (10 points) Let $\alpha \in \overline{F}$ denote a zero of $f(x)$. Write a basis of $E = F(\alpha)$ over F using α. Use the Frobenius automorphism of $F(\alpha)$ to find the other roots of $f(x)$ and express them in this basis. Conclude that $f(x)$ splits over $F(\alpha)$. Are there any intermediate fields E' such that $F < E' < E$?

Problem 4 Galois theory

Let $p > 2$ be a prime number and $f(x) \in \mathbb{Q}[x]$ be a degree p irreducible polynomial over $\mathbb{Q}$. Let K denote the splitting field of $f(x)$.

1. (5 points) Let $\alpha \in \overline{\mathbb{Q}}$ be a zero of $f(x)$. What is the degree of the field extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$?

2. (10 points) The Galois group $G(K/\mathbb{Q})$ can be viewed as a subgroup of the symmetric group S_p. Show that $G(K/\mathbb{Q})$ must contain a cycle of length p. Suppose f has $p - 2$ zeros in $\mathbb{R}$ and two zeros which are complex conjugates. Show that the automorphism of complex conjugation $\sigma : \mathbb{C} \to \mathbb{C}$ restricted to K yields an element $G(K/\mathbb{Q})$ which is a transposition in S_p.

3. (10 points) Conclude that the Galois group $G(K/\mathbb{Q})$ is isomorphic to S_p. Determine for which primes such a polynomial $f(x)$ is solvable by radicals.

   Hint: Show that the symmetric group S_n can be generated by any n-cycle σ and any transposition τ.