Group actions and Burnside's Thm § 16 + 17

Recall:

Def 16.1 Let X be a set and G be a group

An action of G on X is a map $*: G \times X \to X$ s.t.

1. $e * x = x \quad \forall\ x \in X$
2. $(g_1 g_2) * x = g_1 * (g_2 * x) \quad \forall\ x \in X$ and $g_1, g_2 \in G$

Isotropy Subgroups

Def Let X be a G-set. Define

$$X_g = \{x \in X \mid gx = x\} \quad \text{and} \quad G_x = \{g \in G \mid gx = x\}.$$

"elements fixed by $g \in G$" "elements of G fixing x".

Thm 16.12 G_x is a subgroup of G

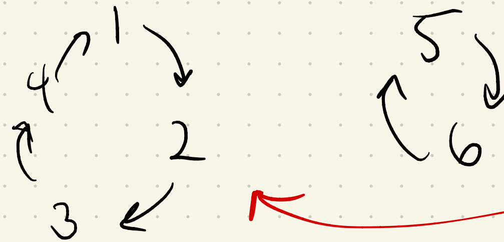
Proof 1) closed: $g_1, g_2 \in G_x$ then $(g_1 g_2)x = g_1(g_2 x) = g_1 x = x$.
$\Rightarrow g_1 g_2 \in G_x$.

2) $ex = x \rightarrow e \in G_x$ 3) If $g \in G_x$ then $x = ex = g^{-1}\underbrace{gx}_{x} = g^{-1}x$.
$g^{-1} \in G_x$.

$\square$

Orbits Recall $\sigma \in S_n$ we considered its cycles.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$$



equivalence classes

"orbits"

Can think of $G = \langle \sigma \rangle$ acting on $X = \{1, \dots, 6\}$.

We can generalise this to any group.

Thm 16.14 Let X be a G-set. For $x_1, x_2 \in X$ define $x_1 \sim x_2 \iff \exists\, g \in G$ s.t. $gx_1 = x_2$.

This is an equivalence relation on X.

Def Let X be a G-set. A cell in the partition of the equivalence relation is an orbit in X under G.

$$\mathcal{O}_x = \{ x' \in X \mid gx = x' \quad g \in X \}.$$

Book writes Gx ⚠

Example: If G acts transitively then $x_1 \sim x_2$ $\forall x_1, x_2 \in X$.

Transitive $\iff$ $Gx = X$ for any x in G.

- Let $G \leq G'$ define an action of G on G' by

$$G \times G' \to G'$$
$$(g, g') \mapsto gg'$$

What are the orbits? left cosets of G

Thm 16.16 Let X be a G-set. Then

$$|\mathcal{O}_x| = (G : G_x) \leftarrow \text{\# of cosets of } G_x \text{ in } G.$$

In particular, if $|G| < \infty$ then $|\mathcal{O}_x|$ divides $|G|$.

Proof

$$\underbrace{\psi : \{G_x, g_1 G_x, g_2 G_x, \ldots\}}_{\text{set of cosets}} \longrightarrow \mathcal{O}_x$$

$$gG_x \longmapsto gx$$

Proceed with caution:

suppose $g' G_x = g G_x$ then $g' = g\hat{g}$ $\hat{g} \in G_x$.

$\psi(g' G_x) = g\hat{g}x = gx = \psi(gG_x)$ well defined on cosets

Ψ is one to one:

$$gx = \Psi(gG_x) = \Psi(g'G_x) = g'x$$

then $g^{-1}g'x = x \Rightarrow g^{-1}g' \in G_x \Rightarrow g' \in gG_x$
$\Rightarrow g'G_x = gG_x$

Ψ is onto:

if $x_1 \in \mathcal{O}_x$ $\exists$ $g \in G$ s.t. $gx = x_1$ Therefore

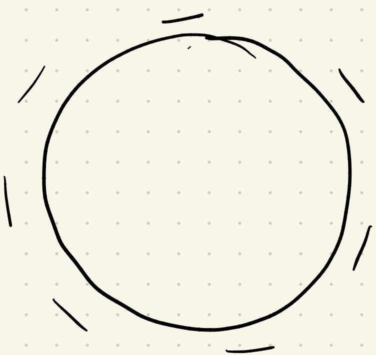
$$x_1 = \Psi(gG_x).$$

$\square$

G-sets and Counting with symmetries § 17

Ex 17.4 How many ways are there to seat n people around a circular table?

$n=7$



1) if table has a distinguished seat then $n!$

2) if not there are $\frac{n!}{n}$.

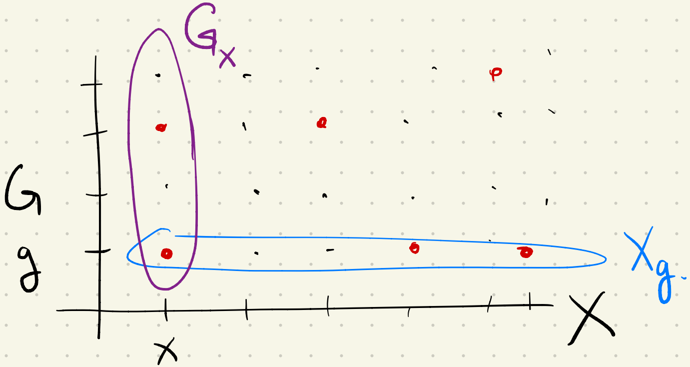
n ways of choosing distinguished seat.

We can formulate this with group actions:

$\mathbb{Z}_n$ acts on $\left\{\begin{matrix}\text{seating arrangements} \\ \text{w/ distinguished chair}\end{matrix}\right\}$ what we count in 2) is # of orbits

Proof By "double counting" (Mat 2250)

$$\mathcal{N} = \{(g, x) \mid gx = x\} \subseteq G \times X$$



partition $\mathcal{N}$ by taking horizontal rows:

$$|\mathcal{N}| = \sum_{g \in G} |X_g|$$

partition $\mathcal{N}$ by taking vertical columns:

$$|\mathcal{N}| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|\mathcal{O}_x|} = |G| \sum_{x \in X} \frac{1}{|\mathcal{O}_x|}$$

If $x' \in \mathcal{O}_x$ then $\mathcal{O}_x = \mathcal{O}_{x'} \Rightarrow |\mathcal{O}_x| = |\mathcal{O}_{x'}|$.

Denoting an orbit by $\mathcal{O}$: $\displaystyle\sum_{x \in \mathcal{O}} \frac{1}{|\mathcal{O}|} = \frac{|\mathcal{O}|}{|\mathcal{O}|} = 1.$

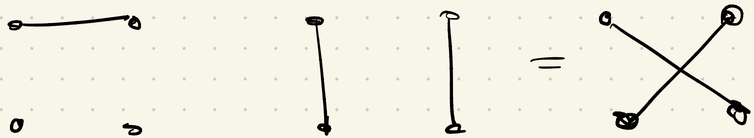
$$\sum_{g \in G} |X_g| = |N| = |G| \left(\sum_{x \in X} \frac{1}{|\mathcal{O}_x|} \right) = |G| \cdot \#\{\text{orbits of } X \text{ under } G\}$$

Challenge Mat 2250:

↗ ways of connecting dots.

How many indistinguishable graphs are there on 4 vertices?

eg.



$X \overset{?}{=}$

$G \overset{?}{=}$

Thm 17.1 (Burnside's formula) Let G be a finite gp and X a finite G-set. Then

$$\left\{\begin{matrix}\#\text{orbits of} \\ X \text{ under } G\end{matrix}\right\} \cdot |G| = \sum_{g \in G} |X_g|$$

recall: $X_g = \{ x \in X \mid gx = x \}$ elts fixed by g.

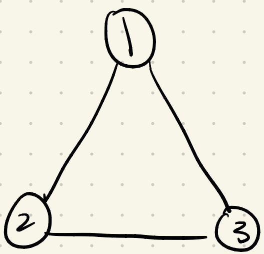
What if we don't distinguish between clockwise or counter-clockwise arrangements? Ie. Beaded necklaces?

$G = D_n$ acts on same set and has different # of orbits.

$$\# \text{necklaces} = \# \text{orbits} = \frac{(n-1)!}{2}.$$

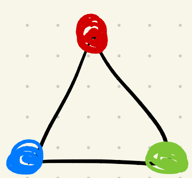
Example 17.6

$\#$ of ways of coloring the vertices of equilateral triangle with 4-colors with repetition allowed.

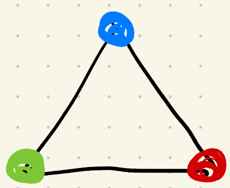


$4^3 = 64$ ways if vertices are labelled.

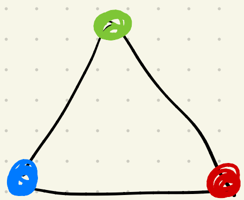
But:



=

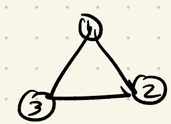


=



These are all the same when vertices are unlabelled.

S_3 is the group acting on colorings of



Orbits are more complicated to count...

Example 17.6

of ways of coloring the vertices equilateral triangle with 4-colors with repetition allowed?

Apply Bernstains Thm:

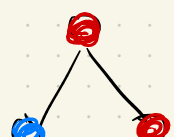
$$\#\left\{\begin{matrix}\text{orbits of} \\ X \text{ under } G\end{matrix}\right\} \cdot |G| = \sum_{g \in G} |X_g|$$

$\underline{S_3}$.

$|X_e| = 64$ any coloring.

$$\#\text{orbits} = \frac{1}{|S_3|} \sum_{g \in S_3} X_g$$

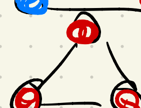
$|X_{(i,j)}| = 4 \cdot 4$



two same color.

$$= \frac{1}{|S_3|} \left(64 + 16 + 16 + 16 + 4 + 4 \right)$$

$|X_{(i,j,k)}| = 4$



monochrome

$$= \frac{120}{6} = 20.$$