Group actions §16

Def 16.1 Let X be a set and G be a group.

An <u>action of G on X</u> is a map $*: G \times X \to X$ s.t.

1. $e * x = x \quad \forall\, x \in X$
2. $(g_1 g_2) * x = g_1 * (g_2 * x) \quad \forall\, x \in X$ and $g_1, g_2 \in G$

Say X is a <u>G-set</u>.

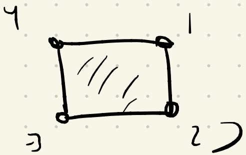
Example. If X is any set then S_X acts on X

$X = \{1, \dots, n\} \qquad * : S_n \times X \to X$

$$(\sigma, i) \mapsto \sigma(i).$$

- Same if $H \leq S_X$ X is still an H-set.

- $X = G$ is a G set $* : G \times G \to G$ $(g_1, g_2) \mapsto g_1 g_2$

- D_4 acts on the square 

also on $\{1, 2, 3, 4\}$,
See example 16.8 for another D_4 set X.
Geir Ellingsrud's note on D_4 acting on the square.

- The Rubic's cube group acts on the Rubic's cube

- Any group G acts trivially on any set X.
$* : G \times X \to X$ $gx = x$ $\forall g \in G$ $x \in X$.

Thm 16.3 Let X be a G-set. For all $g \in G$, the function $\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$ for $x \in X$ is a permutation of X (ie $\sigma_g \in S_X$)

Also $\phi : G \to S_X$, $g \mapsto \sigma_g$ is a homomorphism satisfying

$$\phi(g) \cdot x = gx \quad \forall \; g \in G \quad x \in X$$

Proof See text.

1) Must show σ_g is a bijection (see Cayley's Thm)

2) Show ϕ satisfies homomorphism property.

Question What is the kernel of ϕ?

$\ker\phi = \{ g \in G \text{ s.t. } gx = x \ \forall x \in X \}$.

Def • G acts faithfully on X if $\ker\phi = e$ where $\phi : G \to S_X$.

• A group G acts transitively on a G-set if $\forall \ x_1, x_2 \in X \ \exists \ g \in G$ s.t. $gx_1 = x_2$.

Corollary If G acts on X then $G/\ker\phi = H$ acts faithfully on X via $(aH)x = a \cdot x \quad \forall \ aH \in G/H$.

Task during the break

Find one example of a group action we haven't see.

Isotropy Subgroups

Def Let X be a G-set. Define

$$X_g = \{x \in X \mid gx = x\} \quad \text{and} \quad G_x = \{g \in G \mid gx = x\}.$$

"elements fixed by $g \in G$" "elements of G fixing x".

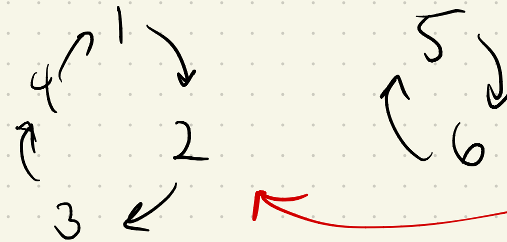
Thm 16.12 G_x is a subgroup of G

Proof 1) closed: $g_1 g_2 \in G_x$ then $(g_1 g_2)x = g_1(g_2 x) = g_1 x = x$.
$\Rightarrow g_1 g_2 \in G_x$.

2) $ex = x \rightarrow e \in G_x$ 3) If $g \in G_x$ then $x = ex = g^{-1}\underbrace{gx}_{x} = g^{-1}x$.
$g^{-1} \in G_x$. $\square$

Orbits. Recall $\sigma \in S_n$ we considered its cycles.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$$



equivalence classes

"orbits"

Can think of $G = \langle \sigma \rangle$ acting on $X = \{1, \dots, 6\}$.

we can generalise this to any group.

Thm 16.14 Let X be a G-set. For $x_1, x_2 \in X$ define $x_1 \sim x_2 \iff \exists\, g \in G$ s.t. $gx_1 = x_2$.

This is an equivalence relation on X.

Def Let X be a G-set. A cell in the partition of the equivalence relation is an orbit in X under G.

$$\mathcal{O}_x = \{ x' \in X \mid gx = x' \quad g \in X \}.$$

Book writes Gx ⚠

Example: If G acts transitively then $x_1 \sim x_2$ $\forall x_1, x_2 \in X$.

Transitive $\iff$ $Gx = X$ for any x in G.

- Let $G \leq G'$ define an action of G on G' by

$G \times G' \to G'$ What are the orbits?

$(g, g') \mapsto gg'$ left cosets of G

Thm 16.16 Let X be a G-set. Then

$$|\mathcal{O}_x| = (G : G_x) \leftarrow \# \text{ of cosets of } G_x \text{ in } G.$$

In particular, if $|G| < \infty$ then $|\mathcal{O}_x|$ divides $|G|$.

Proof

$$\psi : \underbrace{\{G_x, g_1 G_x, g_2 G_x, \dots\}}_{\text{set of cosets}} \to \mathcal{O}_x$$

$$g G_x \mapsto g x$$

Proceed with caution:

suppose $g' G_x = g G_x$ then $g' = g \hat{g}$ $\hat{g} \in G_x$

$\psi(g' G_x) = g \hat{g} x = g x = \psi(g G_x)$ well defined on cosets

Ψ is one to one:

$$gx = \Psi(gG_x) = \Psi(g'G_x) = g'x$$

then $g^{-1}g'x = x \Rightarrow g^{-1}g' \in G_x \Rightarrow g' \in gG_x$

$\Rightarrow g'G_x = gG_x$

Ψ is onto:

if $x_1 \in \mathcal{O}_x$ $\exists$ $g \in G$ s.t. $gx = x_1$ Therefore

$$x_1 = \Psi(gG_x).$$

$\square$

G-sets and Counting §17

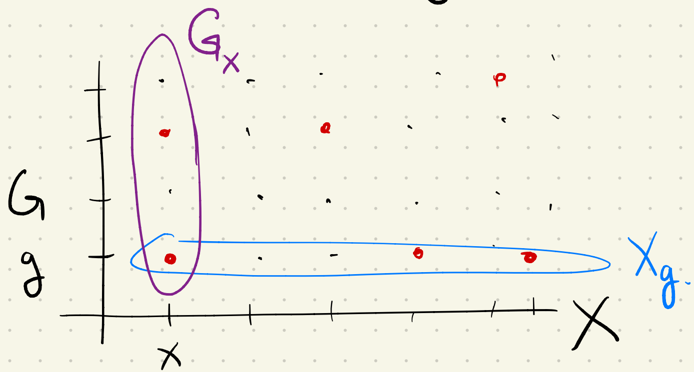
Thm 17.1 (Burnside's formula). Let G be a finite gp and X a finite G-set. Then

$$\#\left\{\begin{matrix}\text{orbits of} \\ X \text{ under } G\end{matrix}\right\} \cdot |G| = \sum_{g \in G} |X_g|.$$

recall: $X_g = \{ x \in X \mid gx = x \}$. elts fixed by g.

<u>Proof</u> By "double counting" (Mat 2250)

$$\mathcal{N} = \{ (g, x) \mid gx = x \} \subseteq G \times X$$



partition $\mathcal{N}$ by taking horizontal rows:

$$|\mathcal{N}| = \sum_{g \in G} |X_g|$$

partition $\mathcal{N}$ by taking vertical columns:

$$|\mathcal{N}| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|\mathcal{O}_x|} = |G| \sum_{x \in X} \frac{1}{|\mathcal{O}_x|}$$

If $x' \in \mathcal{O}_x$ then $\mathcal{O}_x = \mathcal{O}_{x'} \Rightarrow |\mathcal{O}_x| = |\mathcal{O}_{x'}|$.

Denoting an orbit by $\mathcal{O}$:
$$\sum_{x \in \mathcal{O}} \frac{1}{|\mathcal{O}|} = \frac{|\mathcal{O}|}{|\mathcal{O}|} = 1.$$

$$\sum_{g \in G} |X_g| = |N| = |G| \left(\sum_{x \in X} \frac{1}{|\mathcal{O}_x|} \right) = |G| \cdot \#\left\{ \begin{array}{c} \text{orbits of} \\ X \text{ under } G \end{array} \right\}$$