Quiz

1) $(1,2,3)(4,5)(6,7) = \underbrace{(1,3)(1,2)}_{(1,2,3)}(4,5)(6,7) \leftarrow 4$ transpositions

$\Rightarrow$ is an even permutation.

2) $A_5 \leq S_5$ $\quad |A_5| = \dfrac{|S_5|}{2} = \dfrac{5!}{2} = 60$

3) Index of subgroup $\begin{array}{c}\{0,4\} \\ \parallel \\ \langle 4\rangle\end{array} \leq \mathbb{Z}_8$. When $|G| < \infty$

The index of H in G

$[G : H] := \#$ of left cosets of H in G $\underset{\text{Lagrange's Thm}}{\overset{\text{by}}{=}} \dfrac{|G|}{|H|} \Rightarrow \dfrac{|\mathbb{Z}_8|}{|\langle 4\rangle|} = \dfrac{8}{2} = 4$

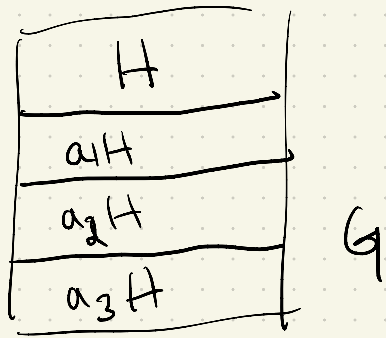
The cosets of $\langle 4 \rangle$ in $\mathbb{Z}_8$ are:

$\langle 4 \rangle = \{0, 4\}$, $1 + \langle 4 \rangle = \{1, 5\}$, $2 + \langle 4 \rangle = \{2, 6\}$, $3 + \langle 4 \rangle = \{3, 7\}$

$\Rightarrow$ There are 4 cosets.

$\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

Lagrange's Thm:



Notice

$\langle 4 \rangle = \{0, 4\} = 4 + \langle 4 \rangle = \{4, 0\}$.

4) Select which ones are true

- Cosets of $H \leq G$ all have same size True!
  Proved $\lambda_6 : H \to aH$ is a bijection

- The right and left cosets aH and Ha of a group H are always equal. FALSE! Example $\langle(1,2)\rangle \leq S_3$

- The coset aH is a subgroup. FALSE!
  aH may not contain the identity. The only coset which is a subgroup is H.

Notice that $H = aH \iff a^{-1} \in H$. Exercise

- The order of a subgroup must divide the order of the group TRUE Lagrange's Thm.

Sets versus Cosets versus Subgroups.

A subgroup H of G is a subset which is
1) closed under multiplication in G.
2) contains $e \in G$ identity
3) $\forall\, a \in H$ we must have $a^{-1} \in H$.

A left coset of H in G is a subset of the form $aH = \{ah \mid h \in H\}$ for some $a \in G$.

A right coset of H in G is a subset of the form $Ha = \{ha \mid h \in H\}$ for some $a \in G$.

Notice that:

$$\{\text{subgroups of } G\} \subsetneq \{\text{cosets of } G\} \subsetneq \left\{\begin{matrix}\text{subsets} \\ \text{of } G\end{matrix}\right\}.$$

- A subgroup H of a group G is a subset of G and also a right and left coset of H in G.
- A left coset aH of H in G is not a subgroup unless $aH = H$. It is always a subset.
- An arbitrary subset $A \subseteq G$ is not necessarily a subgroup nor a coset.

Alternating group $A_n \leq S_n$.

1) What is the index of A_n in S_n?

$[S_n : A_n] :=$ # left cosets of A_n in S_n $= \dfrac{|S_n|}{|A_n|} = \dfrac{n!}{n!/2} = 2.$

2) What are the left and right cosets of A_n in S_n? $S_n = A_n \cup B_n$

left cosets:

$A_n = \{$even permutations$\}$

$\tau A_n = \{$odd permutations$\} = B_n$

τ transposition

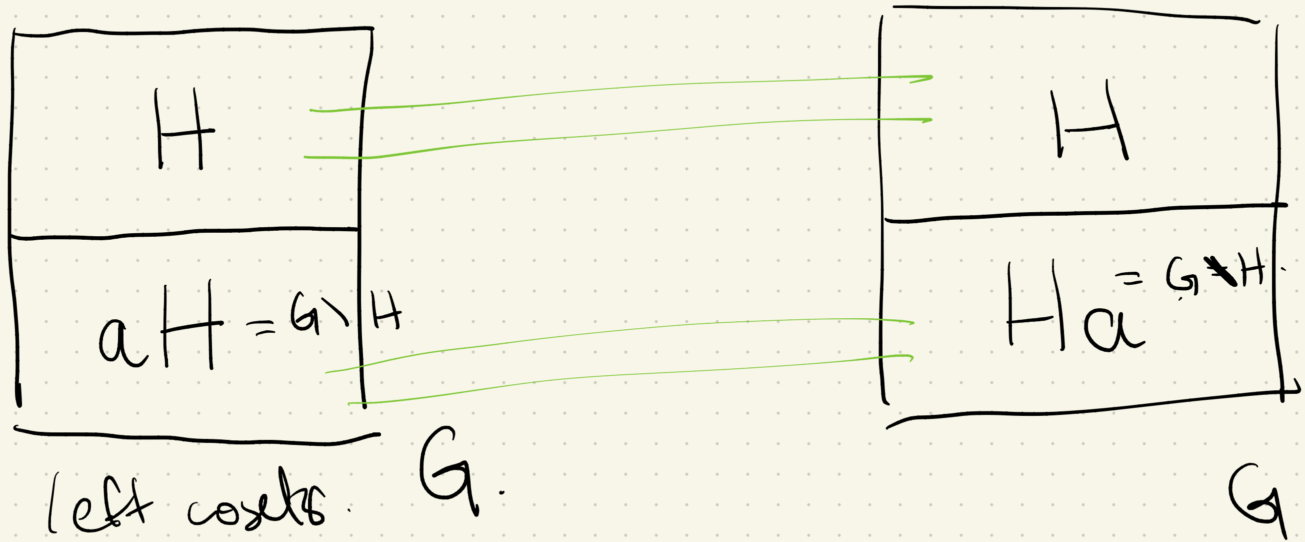
right cosets

$A_n = \{$even perms$\}$.

$A_n \tau = \{$odd perms$\} = B_n$

3) What can one say about left and right cosets of $H \leq G$ when $[G : H] = 2$?

If Index of H in G $= 2$. the right + left cosets are equal.



$$aH = G \setminus H = Ha.$$

Prop If $H \leq G$ has $[G:H]=2$

Then $aH = Ha \quad \forall \; a \in G$.

Prop If G is abelian and $H \leq G$ then

$aH = Ha \quad \forall \; a \in G$.

Foreshadowing Section 14. "Normal subgroups"

Ex. 10.7 - $H = \langle \mu_1 \rangle \leq S_3$. $\mu_1 = (2,3)$ see Exa 8.7

left cosets

$$H = \{\rho_0, \mu_1\},$$
$$\rho_1 H = \{\rho_1\rho_0, \rho_1\mu_1\} = \{\rho_1, \mu_3\},$$
$$\rho_2 H = \{\rho_2\rho_0, \rho_2\mu_1\} = \{\rho_2, \mu_2\}.$$

osets is

right cosets.

$$H = \{\rho_0, \mu_1\},$$
$$H\rho_1 = \{\rho_0\rho_1, \mu_1\rho_1\} = \{\rho_1, \mu_2\},$$
$$H\rho_2 = \{\rho_0\rho_2, \mu_1\rho_2\} = \{\rho_2, \mu_3\}.$$

10.9 Table

	ρ_0	μ_1	ρ_1	μ_3	ρ_2	μ_2
ρ_0	ρ_0	μ_1	ρ_1	μ_3	ρ_2	μ_2
μ_1	μ_1	ρ_0	μ_2	ρ_2	μ_3	ρ_1
ρ_1	ρ_1	μ_3	ρ_2	μ_2	ρ_0	μ_1
μ_3	μ_3	ρ_1	μ_1	ρ_0	μ_2	ρ_2
ρ_2	ρ_2	μ_2	ρ_0	μ_1	ρ_1	μ_3
μ_2	μ_2	ρ_2	μ_3	ρ_1	μ_1	ρ_0

left cosets in multiplication table.

Def (foreshadowing)

A subgroup $H \leq G$ is normal in G if $\forall a \in G \quad aH = Ha$.

Lagrange's Thm If $H \leq G$, then $|H|$ divides $|G|$..

We actually show: If $H \leq G$ then $|G| = [G:H]|H|$

1) What would the converse statement to Lagrange's theorem be?

"If m divides $|G|$, then $\exists$ a subgroup $H \leq G$ of order m."

NOT TRUE IN GENERAL.

2) Find examples where it holds or doesn't

Foreshadowing: Cayley's Thm + Sylow's Thms