17th March, 2022

MAT 2200

Mandatory assignment 1 of 1

Submission deadline

Thursday 31st March 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

You are highly encouraged to collaborate with other students on the problem set. If you need help connecting with others in the course, let me know by email. You must include the names of any students who you collaborated with. Remember: the write up of all solutions must be your own!

Problem 1. Let A be any non-empty subset of a group G. Define

$$C(A) = \{g \in G \mid g^{-1}ag = a \text{ for all } a \in A\}.$$

1. Show that $C(A)$ is a subgroup of G.
2. The automorphism group of G is

   $$Aut(G) = \{f : G \to G \mid f \text{ is an isomorphism}\}$$

   with group operation given by function composition (convince yourself that this is a group). Let $\phi : G \to Aut(G)$ be defined by $\phi(g) = \gamma_g$ where $\gamma_g(h) = ghg^{-1}$. Show that ϕ is a homomorphism. What is its kernel?
3. Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Is the homomorphism $\phi : G \to Aut(G)$ surjective? Justify your answer.

Problem 2. Let G be a finite group acting transitively on the finite set X and suppose $|X| > 1$. Show that there is an element $g \in G$ that fixes no element of X, i.e. there exists a $g \in G$ such that $X_g = \{x \in X \mid gx = x\} = \emptyset$.

Hint: Use that G acts transitively to compare the size of X and G and then apply Burnside's formula.

Problem 3. Let G be a finite group and $\varphi : G \to G$ be a group homomorphism. Let p be a prime number dividing $|G|$.

1. Show that if the order of $g \in G$ is a power of p, then the order of $\varphi(g) \in G$ is also a power of p (not necessarily the same power).
2. Suppose the p-Sylow subgroup P of G is normal. Show that $\varphi(P) \subseteq P$.

   Hint: You may assume Corollary 36.4 from Fraleigh, i.e., the size of a (sub)group is a power of p if and only if every element of the (sub)group is a power of p (not necessarily the same power).

Problem 4. A ring $(R, +, \cdot)$ is a *Boolean ring* if $a^2 = a$ for every $a \in R$.

1. Show that in a Boolean ring, $a + a = 0$ for every element a. (Hint: consider $(a + a)^2$.)

2. Show that a Boolean ring is necessarily commutative. (Hint: consider $(a + b)^2$.)

3. Suppose R is a Boolean ring with more than 2 elements including a unity 1. Show that R can not be an integral domain.