Generating functions for partitions

Recall P_n denotes the number of partitions of a positive integer n

$$P_5 = 7 \quad \{5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1\}$$

We will determine the generating function for P_n:

$$P(z) = \sum_{n \geq 0} P_n z^n.$$

It is easier to begin with

$$P^{dist}(z) = \sum_{n \geq 0} P_n^{dist} z^n$$

where P_n^{dist} is the number of partiti[...]

of n with distinct par...

$P_5^{dist} = 3 \quad \{5, 4+1, 3+2\}$

Consider the product

$$\prod_{i=1}^{m} (1+z^i) = (1+z)(1+z^2)\cdots(1+z^m)$$

$$= 1 + z + z^2 + (z^3 + z^2 z^1) + (z^4 + z^3 z^1)$$
$$+ (z^5 + z^4 z^1 + z^3 z^2) + (z^6 + z^5 z^1 + z^4 z^2) + \cdots$$

We see z^K appears once for every partition of K into destinct parts for $K \leq m$.

Therefore

$$\sum_{n\geq 0}^{m} P_n^{dist} z^n = \prod_{i=1}^{m} (1+z^i)$$

and to get the infinite formal

Series we take an infinite produ

So $$P^{dist}(z) = \prod_{i=1}^{\infty} (1+z^{i})$$

Recall we ignore issues of convergence so the infinite product causes no problems and to find P_K^{dist} we only need a product of K terms!

To obtain $P(z)$ we need represent repeated parts:

$$P(z) = \prod_{i=1}^{\infty} (1 + X^{i} + X^{2i} + X^{3i} + \dots\dots)$$

$$= (1+X+X^{1+1}+\dots)(1+X^{2}+X^{2+2}+\dots)(1+X^{3}+X^{3+3}+\dots)$$
$$\cdot (1+X^{4}+X^{4+4}+\dots)$$

Again to see why this is true we calulate the product in low degrees

$$= 1 + x + (x^{1+1} + x^2) + (x^{1+1+1} + x^2x^1 + x^3) + (x^{1+1+1+1} + x^2 x^{1+1} +$$
$$x^{2+2} + x^{3+1} + x^4) + \dots$$

Each term producing x^k is in bijection with a partition of k for low enough degree.

So

$$P(z) = \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + \dots) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

using that:

$$\frac{1}{1-x^i} = (1 + x^i + x^{2i} + \dots).$$

If we want the generating function

for partitions with no parts equal to k we simply take:

$$\prod_{\substack{i=1 \\ i \neq k}}^{\infty} (1 + x^i)$$

Recall we also considered the number of partitions of n into exactly k parts $P_{n,k}$.

A generating function for $P_{n,k}$ uses two variables z, w.

$$Q(z, w) = \sum_{n,k \geq 0} P_{n,k} z^n w^k .$$

Notice that

$$Q(z,w) = \prod \frac{1}{(1 - w z^i)}$$

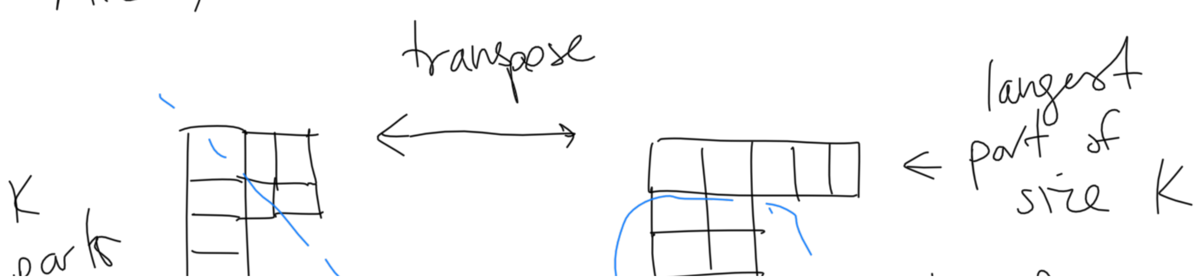
And $Q(z,1) = P(z)$.

For K fixed, the coefficient of w^K in $Q(z,w)$ gives the generating function for # of partitions with K parts.

Recall from exercise 1.11.

$$P_{n,k} = P_{n-k,1} + \text{—} + P_{n-k,k}$$

Also,



partition of $n-k$.

$$\sum_{n\geq 0} P_{n,k}\, z^n = \frac{z^k}{(1-z)(1-z^2)\cdots(1-z^k)}$$

$$\frac{z^k}{1-z^k} = z^k + z^{2k} + \cdots\cdots$$

every term in the product must have a factor of $z^k \Rightarrow$ every partition being counted has largest part equal to k.

(this corresponds to conjugate/transpo[se] partitions).

Notice we obtain the relation

$$Q(z,w) = \prod_{i=0}^{\infty} \frac{1}{(1-wz^i)} = \sum \left(\frac{z^k}{(1-z)\cdots\cdots(1-z^k)} \right) w$$

$k \geq 0$

This is what is know as a <u>partition</u> <u>identity</u>.

Recall also P_n^{odd} # of partitions of n with odd parts.

$$P^{odd}(z) = \sum_{n \geq 0} P_n^{odd} z^n$$

$$= \prod_{i=0}^{\infty} \frac{1}{(1 - z^{2i+1})}$$

Last time using Inclusion-Exclusion we proved:

<u>Euler's formula</u>

$$P^{odd}(z) = P^{dist}(z)$$

Proof

$$P^{odd}(z) = \prod_{i=0}^{\infty} \frac{1}{(1-z^{2i+1})} \cdot \prod_{i=0}^{\infty} \frac{(1-z^{2i})}{(1-z^{2i})}$$

$$= \prod_{i=0}^{\infty} \frac{(1-z^{2i})}{(1-z^{i})}$$

And $\frac{(1-z^{2i})}{(1-z^{i})} = 1 + z^{i}$ so

$$P^{odd}(z) = \prod_{i=0}^{\infty} (1+z^{i}) = P^{dist}(z)$$

$\square$

Can you come up with a

bijective proof ??