Fundamental counting coefficients
(Sections 1.1, 1.2, 1.3).

Binomial coefficients

$N = \{1, \ldots, n\}$ size n-set

Def.

$$\binom{n}{k} := \text{\# of size } k \text{ subsets of an } n\text{-sized set}$$

Multiplicative formula

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

(~ 850 A.D.

Recursive / Summation formula

$$\binom{n}{} \quad \binom{n-1}{} \quad \binom{n-1}{}$$

$$\binom{n}{k} = \underbrace{\binom{n-1}{k}}_{\text{subsets not containing } i} + \underbrace{\binom{n-1}{k-1}}_{\text{subsets containing } i}$$

For $i \in N$

For the above recurrence to determine $\binom{n}{k}$ we need also

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n} = 1$$

<u>Exercise</u>: $\binom{n}{k}$ = # of binary sequences of length n with exactly k 0's.

Find a bijection between sequences and subsets ("law of equality" in text.)

This description and recursive formula for $\binom{n}{k}$ date back to ~2 AD

Pingala (sanskrit poet) enumeration
meters with short & long syllables
First use of zero (as a place
holder) also possible first occurren
of the fibonnaci sequence
$f_n = f_{n-1} + f_{n-2}$ $f_0 = 1$ $f_1 = 1$ (1202)

Pascal's triangle (1650)
Arrange $\binom{n}{k}$ into a
triangle form "staircase of
Mont Meru

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Exercise (1.14 book) Show the sum along diagonals of the triangle are the Fibonnaci #'s

Exercise (Fun) Draw Pascal's triangle staircase Mont Meru modulo 2 (binary)

$0+0=0$, $0+1=1$, $1+0=1$, $1+1=0$

What fractal shape does it resemble?

What rows are all 0's except for 1st and last entries?

... multiplication formula

We can prove multiplicative formula

by considering

k-permutations of N set = words of length k with no repetitions from alphabet N.

"words" like "lists" like "sequences" differ from sets in that <u>order</u> <u>matters</u>.

$$= n(n-1)\cdots(n-k+1) =: n$$

falling factorial

$$\binom{n}{k} = \frac{\#\,k\text{ perms of } n}{k!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

Counting partitions

Def Stirling #'s of the second kind are:

$$S_{n,k} := \text{\# of set partitions of } N \text{ into } k \text{ pieces}$$

(Recall $|N| = n$).

Eg. $S_{5,2} = 10$ see page 10.

$S_{n,n} = 1$ $\qquad$ $S_{n,1} = 1$

$S_{0,0} = 1$ $\qquad$ $S_{n,0} = 0.$

Proposition

$$S_{n,k} = k\, S_{n-1,k} + S_{n-1,k-1}$$

Proof consider $i \in N$.

partitions in which i is a singleton $= S_{n-1}, S_{k-1}$

partitions where i is not a singleton $= k S_{n-1,k}$.

$$\Rightarrow S_{n,k} = k S_{n-1,k} + S_{n-1,k-1}. \quad \square$$

Exercise

$$S_{n,n-1} = \binom{n}{2}$$

$$S_{n,2} = 2^{n-1} - 1$$

prove using recurrence or <u>bijection</u>

Integer partitions

Def $P_{n,k}$ = # of int partitions of n with k summ

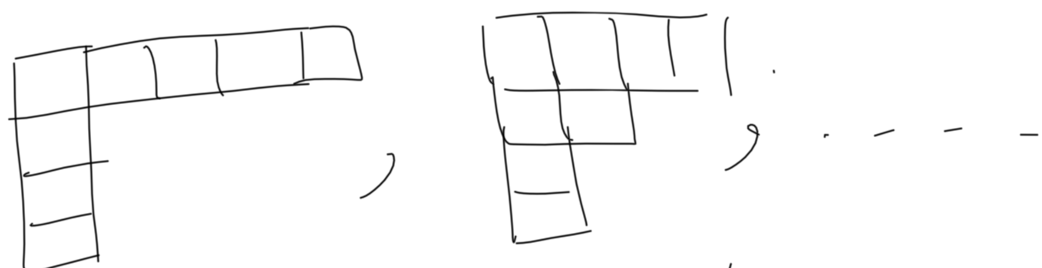
$p_{n,k}$ of n into k summ

$P_{5,2} = 2$ $\quad 5 = 4+1 \quad 5 = 3+2$

k-rows
n-boxes

Young diagrams

$P_{8,4} = 5$ see list on page 11 ie.



$5+1+1+1$ $\quad$ $4+2+1+1$

Hard to count !!

Ordered integer partitions easier to count.

Ordered 2-partitions of 5.

$\{4+1, 1+4, 2+3, 3+2\}$

$= 4.$

Ordered 4 partitions of 8.

$$35 = \underbrace{4 + 12 + 6 + 12 + 1}$$

correspond to distinct reorderings of partitions listed in text.

$5+1+1+1,\ 1+5+1+1,\ 1+1+5+1,$
$1+1+1+5$

$2+2+2+2$

Proposition

$$\#\text{ordered } k \text{ partitions of } n = \binom{n-1}{k-1}.$$

Proof via bijection (rule of equality)

$$\left\{ \begin{array}{c} \text{ordered } k \\ \text{partitions} \\ \text{of } n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} k-1 \text{ subsets} \\ \text{of size } k \end{array} \right\}.$$

$$n = n_1 + \ldots + n_k \longrightarrow \{n_1, n_1+n_2, \ldots\ldots n_1 + \ldots + n_{k-1}\}$$

Inverse mapping is then

$$\{a_1, \ldots, a_{k-1}\} \to a_1 + (a_2 - a_1) + \ldots\ldots + (n - a_{k-1}).$$

$$= \underline{n}.$$

$\square$.

Multi-sets + Multiset partitions

Multisets are sets with repetitio[n] allowed.

$M = \{1, 1, 2, 3, 3, 4\}$ is a multi[set] on $N = \{1, 2, 3, 4\}$.

Proposition

of k-sized multisets over an n-set $= \dfrac{n(n+1)\dots(n+k-1)}{k!}$

$= \dfrac{n^{\overline{k}}}{k!} = \dbinom{n+k-1}{k}$

"rising factorials"

Proof. bijection.

$$\left\{\begin{matrix}k \text{ multiset} \\ \text{over } N\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}k\text{-subsets} \\ \text{of } \{1,\dots,n+k-1\}\end{matrix}\right\}.$$

$$\{a_1 \le a_2 \le \dots \le a_k\}$$

$$\downarrow$$

$$\{a_1, a_2+2, a_3+3, \dots, a_k+(k-1)\}.$$

$\{1$ $b_k\}$

$$\downarrow$$

$$\{b_1, b_2 - 1, \dots, b_k - (k-1)\}$$

reorder if you need.

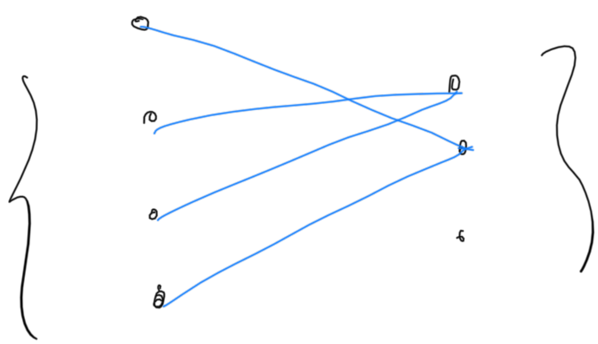
Mappings between sets.

(A viewpoint useful for Ramsey theory, graph theory, + relations between coefficients).

$f : N \longrightarrow R \qquad |N| = n \quad |R| = r$ finite sets.

a map

N



R.

$$\#\{f : N \to R\} = r^n$$

$$\# \{ f : N \to R \mid f \text{ injective} \}$$

$$= r(r-1)\ldots(r-n+1)$$

$$= r^{\underline{n}}$$

$$\# \{ f : N \to R \mid \text{surjective} \}$$

$$= \underbrace{r! \underbrace{S_{n,r}}_{\text{unordered set partitions}}}_{\text{ordered set partitions}}$$

$$\underbrace{f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(r).}_{\text{non-empty sets.}}$$

Proposition

$$r^n = \sum_{k=0}^{n} S_{n,k} r^k$$

$$r^n = |\mathrm{map}\{N,R\}| = \sum_{A \subset R} |\mathrm{surj}(N,A$$

$$= \sum_{k=0}^{r} \sum_{|A|=k} (\mathrm{surj}(N,A))$$

$$= \sum_{k=0}^{r} \underbrace{\binom{r}{k}}_{\#\ k \text{ subsets } A \subset R.} k!\, S_{n,k}$$

$$= \sum_{k=0}^{r} r^{\underline{k}} S_{n,k} \qquad \square.$$