Graph Theory continued

Feb 18^{th}

(Chapter 6)

Representation of Graphs.

Equivalent non-pictoral ways of representing a graph.

Let $G = (V, E)$ be a graph

The adjacency matrix of G is an $n \times n$ matrix $(n = |V|)$

$A_G = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1 & u_i u_j \in E \\ 0 & u_i u_j \notin E. \end{cases}$$

The incidence matrix $B_G = (b_{ij})$ is a $n \times q$ matrix $q = |E|$ with

$$b_{ij} = \begin{cases} 1 & \text{if } u_i \in k_j \\ 0 & u_i \notin k_j \end{cases}$$

Notice that A_G is symmetric.

Also

$$B_G B_G^T = \begin{pmatrix} d(u_1) & & \\ & \ddots & \\ & & d(u_n) \end{pmatrix} + A_G.$$

Complete graph $K_n = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$

$n \times n$ matrix of ones off the diagonal

For G a bipartite graph on $V = T \cup S$
$|T| = n$ $|S| = m$
we obtain

$$A_G = \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{pmatrix} 0 & \boxed{D} \\ \boxed{D} & 0 \end{pmatrix} \end{array}$$

where $D_G = (d_{ij})$ is given by $d_{ij} = \begin{cases} 1 & u_i v_j \in E \\ 0 & \end{cases}$.

<u>Example</u>. For a graph G.

$(A_G^{\ell})_{ij} = $ # of paths in G from u_i to u_j of length ℓ.

For $\ell = 1$ this is def of A_G.

Continue by induction:

$$A_G^{l}(i,j) = \sum_{k=1}^{n} \underbrace{A_G^{l-1}(i,k)}_{\substack{\#\text{ paths of}\\ \text{length } l-1\\ \text{ending at some}\\ k}} \underbrace{A_G(k,j)}_{\substack{\#\text{ ways}\\ \text{of proceed}\\ \text{to } j}}$$

Consider the last stop of a path of length l before arriving at j (let it be k). Then the summation above is just the rule.

Different labellings of a graph yield different adjacency matrices (differ by permutations of rows + columns).

Def The bandwidth of a graph $G = (V, E)$ with vertices labelled by $f : V \to \{1, \dots, n\}$.

is defined by

$$b_f = \max_{u,v \in E} |f(u) - f(v)|.$$

The bandwidth of G is

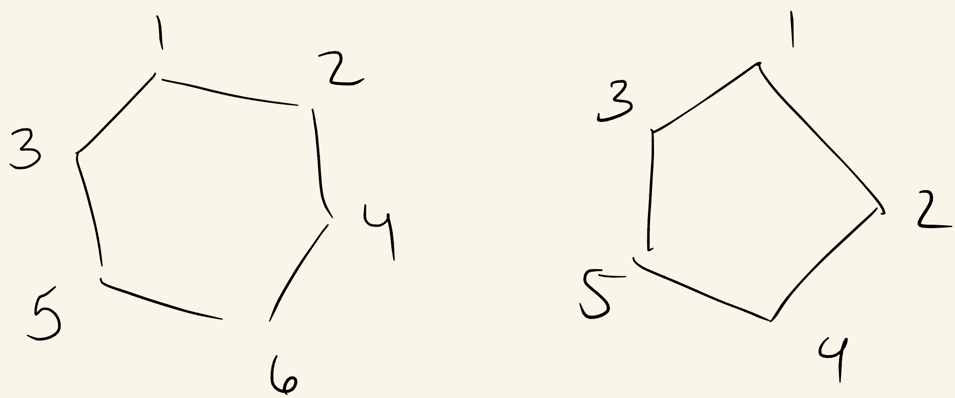
$$b(G) := \min_{f \text{ labellings}} b_f \geq 1.$$

The path P_n with labelling

has bandwidth $= 1$.

This is the only graph with non-isolated vertices with $b(G) = 1$?

What about a circuit C_n?



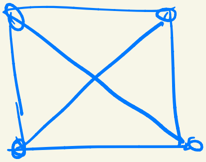
$$b(K_n) = b_f(K_n) = n-1 \text{ for}$$

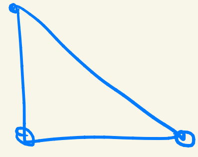
any labelling f.

Paths, circuits, and subgraphs.

Def A graph $H = (V', E')$ is a subgraph of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The subgraph H is an <u>induced</u> subgraph if $E' = E \cap \binom{V'}{2}$.

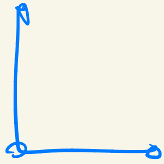
(ie. the edges of H are precisely the edges in G between vertices in V').





H

induced



H'.

not induced

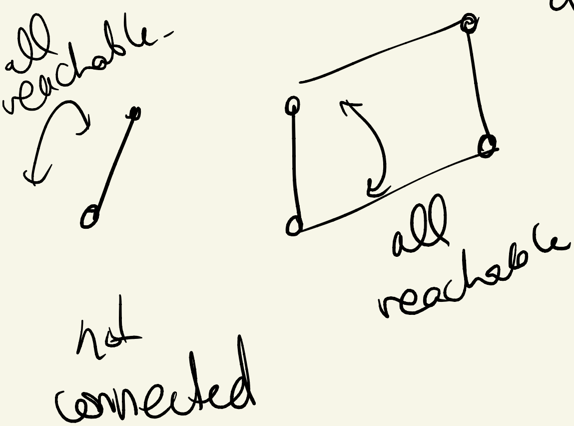
Examples of subgraphs :

- $G \setminus A$ denotes induced subgraph on $E \setminus A$ $\quad A \subseteq V$.
- $G \setminus B$ denotes $G = (V, E \setminus B)$. $B \subset E$.

<u>Def</u> $G = (V, E)$ then $v \in V$ is reachable from $u \in V$ if $\exists$ a path P in G from u to v.

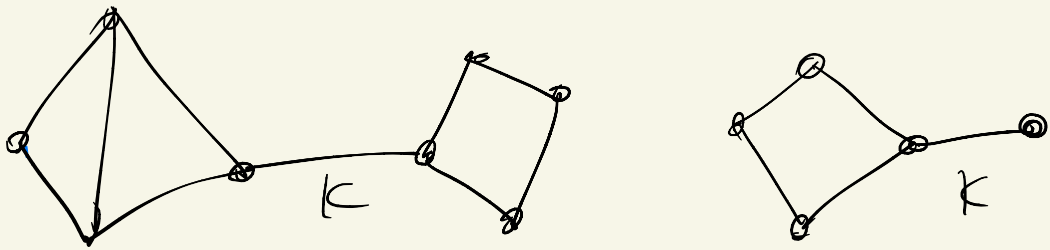
Reachability defines an equivalence relation on V.

Equivalence classes are <u>connected components</u> of G



G is <u>connected</u> if it consists of a single connected component.

An edge K is a <u>cut edge</u> or a <u>bridge</u> if its removal $G \setminus K$ has more connected component than G.



cut edges K.

The vertex set of G has a distance function

$$d(u,v) = \begin{cases} \min |P| & P \text{ path } u,v \\ \infty & \text{no path } u,v. \end{cases}$$

Exercise:

Verify that $d(u,v)$ is a distance function. (triangle inequality).

The diameter of a graph G is $d(G) := \max_{u,v \in V} d(u,v)$.

Example. $d(Q_n) = n$. since

$d(u,v) =$ # coordinates of u,v that are different.

$d((0\ldots0),(1\ldots1)) = n.$

Thm 6.4 (As stated in book is false!)

A graph G with $n \geq 2$ vertices is bipartite iff all circuits are even length. In particular, G is bipartite iff there are no circuits of odd length.

Proof. May assume G is connected.

If G is bipartite any path must alternate between S and T. i.e. must have an even # of steps to start at S and end at S.

Conversely, suppose all circuits have even length. We will

Construct a partition $V = S \sqcup T$.

Choose a $u \in V$ and set

$$v \in \begin{cases} S & d(u,v) \text{ is even} \\ T & d(u,v) \text{ is odd.} \end{cases}$$

We must show that there are edges between elements of S (neither between T).

Suppose $v, w \in T$ and $\{v, w\} \in E$

Since $uw \in E$ $\quad |d(u,v) - d(u,w)| \leq 1$.

by the triangle inequality but then $d(u,v), d(u,w) = 0$ since they have the same parity.

Let P be a u,v path of length $d(u,v)$

and P' a u,w path of length $d(u,w)$ and x the last common vertex of P and P'.

Then $d(x,v) = d(x,w)$ and we can build a circuit

$$P(x,v),\ vw,\ P'(w,x)$$

of odd length which is a contradiction

<u>Example</u> Q_n is bipartite for all n.

$$S = \{ u \mid \#\ 1\text{'s is even} \}.$$

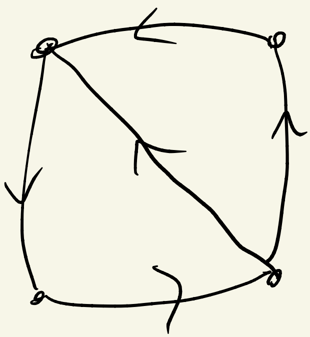
$$T = \{ v \mid \#\ 1\text{'s is odd} \}$$

Directed (oriented) graphs.

In a directed graph edges have an orientation

Therefore edges are denoted (u,v) instead of $\{u,v\}$.

Pictorally draw arrows on edges



Def $\vec{G} = (V, E)$ a directed graph consists of a vertex set V and an edge set $E \subset V \times V \setminus \Delta := \{(u,u) \in V \times V\}$.

For $k = (u,v) \in E$ call u start vertex and v end vertex.

Every directed edge appears at most once in a directed graph also do not allow loops.

For a directed graph we can define

$$d^{+}(u) = \left|\{ k \in E \mid k^{+} = u \}\right| \leftarrow \text{In degree}$$

$$d^{-}(u) = \left|\{ k \in E \mid k^{-} = u \}\right| \leftarrow \text{out degree}$$

Notice

$$\sum_{u \in V} d^{+}(u) = \sum_{u \in V} d^{-1}(u) = |E|$$

The incidence matrix $B = (b_{ij})$ is the $n \times q$ matrix

$$b_{ij} = \begin{cases} 1 & \text{if } u_i = k_j^+ \\ -1 & \text{if } u_i = k_j^- \\ 0 & \text{if } u_i \notin k_j \end{cases}$$

Notice the columns sum to zero and the i^{th} row sum is $d^+(u_i) - d^-(u_i)$.

$$BB^T = \begin{pmatrix} d(u_1) & & \\ & \ddots & \\ & & d(u_n) \end{pmatrix} - A$$

where A is the adjacency matrix

of the underlying undirected multi-graph

A directed path in $\vec{G}$ is a sequence of vertices $u_1 \dots u_n$ in V s.t. $u_i \to u_{i+1}$ are edges.

A directed circuit in $\vec{G}$ is a sequence

$$u_1 \to u_2 \to \dots \to u_n \to u_1$$

A graph $\vec{G}$ is acyclic if there are no directed circuits.

Every graph G can be equipped with orientation to be an acyclic directed graph $\vec{G}$.

Proposition Suppose $\vec{G}$ is acyclic then there exists a source and a sink vertex

$\{u \text{ s.t. } d^-(u)=0\}$ $\{u \text{ s.t. } d^+(u)=0\}$.

Proof Let P be a maximal directed path in $\vec{G}$ from u to v. If (w,u) then w would have to be on the path P otherwise we could add it and P would not be maximal. But then

$$\begin{tikzcd} u \arrow[r] & \cdots \arrow[r] & w \arrow[r] \arrow[d] & \cdots \arrow[r] & v \\ & & u & & \end{tikzcd}$$

This is a directed cycle

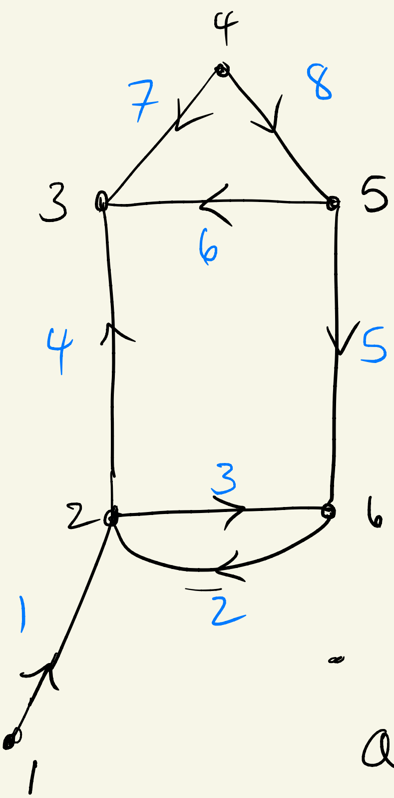
So $\vec{G}$ is not acyclic.

Therefore u is a "source".

Similar proof shows existence of "sinks".

Def A directed graph $\vec{G}$ is strongly connected if there exists a directed path from every vertex u to every other vertex v.

Example



$$B = \begin{matrix} -1 & 0 & \cdots & & & & & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \end{matrix}$$

- Is the graph acyclic?
- Sources and sinks?

Sources 1, 4

sinks 3

Maze tour algorithm

<u>Goal</u> starting from a vertex u_0 want to traverse all edges of G in each direction exactly once and return to u_0.

1) No edge can be traversed in the same direction more than once

2) When we reach $v \neq u_0$ for the first time we mark the edge (u,v) that led us to v. On leaving v, we are allowed to traverse a marked edge (u,v) only after all edges

(v, x) $x \neq u$ have been

Proof that the algorithm gives a "maze tour"

Let $u_0 \to \dots \to u_p = W$ be a tour through the edges given by the algorithm.

We have $u_p = u_0$ and $d_W^+(u) = d_W^-(u)$ (each time we arrive at u we also leave).

We want to prove that $\forall$ $u \in G$ $d_W^+(u) = d_W^-(u) = d_G(u)$

For u_0 this holds.

Suppose v is the 1st vertex not satisfying $*$ in W.

Then by rule (2) the marked edge (u,v) has not yet been used in direction (v,u) for some (u,v) where u proceeds v at some point in W.

But then the predecessor u in W also has $d^{+}_{W}(u) = d^{-}_{W}(u) < d(u)$ contradicting that v was first. Therefore every vertex has

$$d^{+}_{W}(u) = d^{-}_{W}(u) = d_{W}(u).$$