Minimal spanning trees

Def A weighted graph is a graph $G=(V,E)$ together with a weight function

$$w: E \to \mathbb{R}.$$

Question How to find a spanning tree of minimal weight?

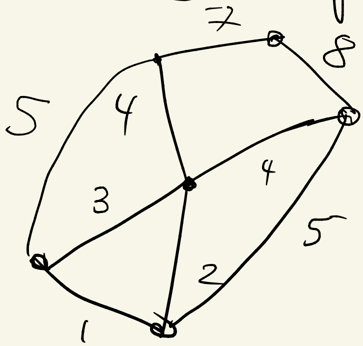
$$w(T) = \sum_{k \in E(T)} w(k)$$

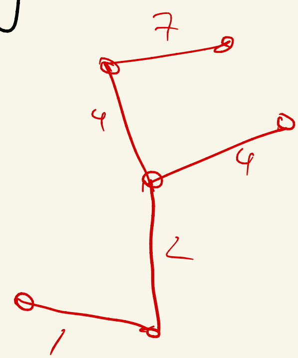
Greedy algorithm Suppose $\{K_1 \ldots K_\ell\}$ have been choosen.

1) choose $K_{\ell+1}$ of minimal weight in $E \setminus \{K_1 \ldots K_\ell\}$ so that $K_1, \ldots, K_{\ell+1}$ contains no circuit.

Repeat Step 1 $n-1$ times

Claim $T = \{K_1 \ldots K_{n-1}\}$ is a minimal spanning tree.





To see why this algorithm works we want to analyse the properties of subforests of graphs.

Let $G = (V, E)$ be a graph

$$\mathcal{W} = \{ F \mid F \subset G \text{ is a forest} \}.$$

The pair $(E, \mathcal{W})$ is a <u>matroid</u>

<u>Def 7.11</u> (Matroid)

Let S be a finite set and $\mathcal{U} \subseteq \mathcal{B}(S)$ be a family of subsets of S.

The pair $\mathcal{M} = (S, \mathcal{U})$ is called a <u>matroid</u> and $\mathcal{U}$ the family of <u>independent</u> sets of $\mathcal{M}$ if

1) $\emptyset \in \mathcal{U}$
2) $A \in \mathcal{U}$, $B \subset A \Rightarrow B \in \mathcal{U}$
3) $A, B \in \mathcal{U}$, $|B| = |A| + 1 \Longrightarrow$
$\exists\, v \in B \setminus A$ s.t. $A \cup \{v\} \in \mathcal{U}$.

A maximal (by inclusion) independent set is called a <u>basis</u>.

<u>Lemma</u> If B_1, B_2 are bases of $\mathcal{M}$ then $|B_1| = |B_2|$.

Suppose $|B_1| < |B_2|$. We can find $A \subseteq B_2$ s.t. $|A| = |B_1| + 1$ moreover $A \in \mathcal{U}$ by (2). Therefore by (3) $\exists$ $v \in A \setminus B_1$ s.t. $B_1 \cup v \in \mathcal{U}$. Hence B_1 was not maximal $\Rightarrow$ B_1 not a basis and we have a contradiction. Therefore $|B_1| = |B_2|$. $\square$

"Matroid" comes from matrices. Abstraction of the notion of independence in mathematics.

let $S = \{v_1, \dots, v_q\}$

where $v_i \in \mathbb{R}^m$ vectors.

then $\mathcal{U} = \{\{v_{i_1} \dots v_{i_k}\} \mid v_{i_1} \dots v_{i_k}$ linearly indp$\}$.

Then $(S, \mathcal{U})$ is a matroid. Axiom 1 + 2 are clear

Axiom 3 Steinitz exchange axiom from linear algebra.

Proposition let G be a graph (V, E) and equip each edge with an orientation.

Let $S = \{ V_K \mid V_K$ is a column of $\vec{B} \}$

($\vec{B}$ oriented incidence matrix)

Then the elements of $\mathcal{U}$ correspond to elts of $\mathcal{W}$.

Ie. $V_{K_1} \dots V_{K_e}$ linearly independent $\iff$ $K_1 \dots K_e$ is a forest in G.

Corollary $M = (E, \mathcal{W})$ for a graph $G = (V, E)$ is a matroid.

To not just rely on linear algebra + to practice graph theory techniques we will prove this directly.

Thm 7.12 If $G = (V, E)$ is a graph, then $M = (E, \mathcal{W})$ is a matroid

Proof. Axiom 1 is satisfied by def.

Axiom 2: If $F \in \mathcal{W}$ then F contains no circuit so $F' \subset F$ also contains no circuit so $F' \in \mathcal{W}$.

Axiom 3 Let F, F' be both forests with $|F'| = |F| + 1$

Since both F and F' are forests we have:

$$|F| = |V| - \#\text{components in } F$$

$$|F'| = |V| - \#\text{components in } F'.$$

$|F'| = |F| + 1 \Rightarrow F'$ has one less component then F

If $(V_1, F_1) \ldots (V_t, F_t)$ are the components of F then F' can have at most $|F_i|$ edges on V_i otherwise we have a circuit in F'. $\Rightarrow$ there exists a K in F' which connects two V_i, V_j

So $W'' = (V, F \cup K)$ is a forest axiom 3 is satisfied. $\square$

<u>Theorem</u> (Nelson 2018)

"Almost all matroids are <u>non</u>-<u>representable</u>"

Do not come from graphs or vector configurations.

As $n = |S| \to \infty$ the proportion of n-element matroids $\to 0$.

Nonetheless matroids characterise when the Greedy algorithm works.

Theorem 7.13 Let $\mathcal{M} = (S, \mathcal{U})$ be a matroid with weight function $w : S \to \mathbb{R}$. The greedy algorithm produces a basis of minimal weight:

1) Let $A_0 = \emptyset \in \mathcal{U}$.

2) If $A_i = \{a_1, \dots, a_i\} \subseteq S$ then let $X_i = \{x \in S \setminus A_i : A_i \cup \{x\} \in \mathcal{U}\}$

If $X_i = \emptyset$ then A_i is the desired basis. Otherwise choose an a_{i+1} in X_i of minimal weight and set

$A_{i+1} = A_i \cup \{a_{i+1}\}$. Repeat 2).

Proof let $A = \{a_1, \dots, a_r\}$ be the obtained set. It follows from axiom 3 that A is a basis.

Moreover $w(a_1) \leq w(a_2) \leq \dots \leq w(a_r)$

$w(a_1) \leq w(a_i)$ since $w(a_1)$ has minimal weight among independent elements. For $2 \leq i \leq r-1$

Since $\{a_1, \dots, a_r\} \in \mathcal{U}$ we have $\{a_1, \dots, a_{i-1}\} \in \mathcal{U}$ by axiom 2

Therefore $a_i, a_{i+1} \in X_{i-1}$

and since we chose a_i over a_{i+1}

$w(a_i) \leq w(a_{i+1})$.

Suppose $\exists$ B basis s.t $\omega(B) < \omega(A)$. Assume $\omega(b_1) \leq \ldots \leq \omega(b_r)$. Then there exists a smallest index i s.t. $\omega(b_i) < \omega(a_i)$ and $i \geq 2$.

Both $A_{i-1} := \{a_1 \ldots a_{i-1}\} \in \mathcal{U}$.
$B_i := \{b_1 \ldots b_i\}$

By axiom 3, there would exist a $b_j \in B_i \setminus A_{i-1}$ with $A_{i-1} \cup \{b_j\} \in \mathcal{U}$ ie. $b_j \in X_{i-1}$ and $\omega(b_j) \leq \omega(b_i) < \omega(a_i)$ and the greedy algorithms would have picked b_j over a_i

We have a contradiction $\square$.

The greedy algorithm applied to graphs is called <u>Krusal's</u> <u>algorithms</u>.

Exercise 7.30 asks for the converse

<u>Thm</u> Let $(S, \mathcal{U})$ be a collection of sets satisfying axioms 1 and 2. Show $(S, \mathcal{U})$ is a matroid if and only if the greedy algorithm yields the optimum maximal (by inclusion) set of $\mathcal{U}$ for <u>every</u> weight function

$$w: S \to \mathbb{R}.$$

The computational complexity of the greedy algorithm requires an analysis of sorting algorithms. Postpone until Chapter 9.

Dijkstra's algorithm/shortest path in a graph 7.4.

Let $G=(V,E)$ be connected and with weight function

$$w: E \to \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$$

Let $u, v \in P$ and let P be a path from u to v

$$\ell(P) = \sum_{k \in E(P)} w(k)$$ the

weighted length of P.

$$d_w(u,v) := \min_{P \, u \text{ to } v} \ell(P)$$

we want the path which minimizes the distance.

Fix u. Dijkstra's algorithm returns a spanning tree T whose unique path u to v is the shortest path from u to v.

1) Let $u_0 = u$ $V_0 = \{u_0\}$ $E_0 = \emptyset$
$\ell(u_0) = 0$.

2) Suppose $V_i = \{u_0, u_1, \dots u_i\}$

$E_i = \{k_1, \ldots, k_i\}$ If $i = n-1$ we are done. Otherwise consider for edges $k = vw$ s.t. $v \in V_i$ and $w \in V \setminus V_i$ the expression

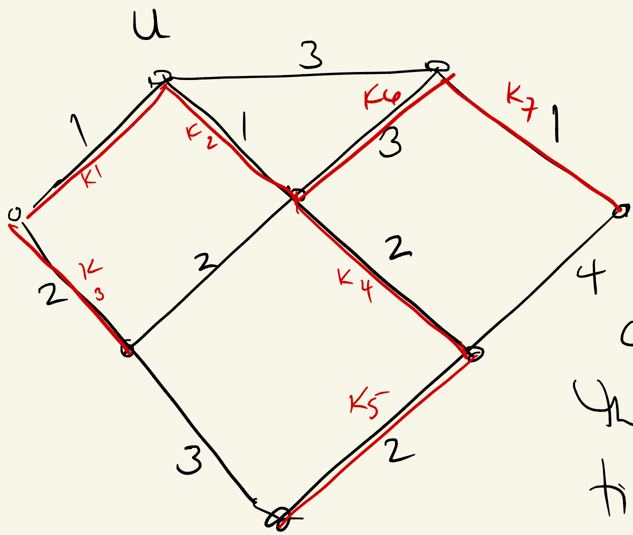
$$f(k) = \ell(v) + \omega(k)$$

where $\ell(v)$ is distance in tree (V_i, E_i) from u to v.

choose $\bar{v}\bar{w} = \bar{k}$ s.t. $f(\bar{k}) = \min f(k)$

Then set $u_{i+1} = \bar{w}$ $k_{i+1} = \bar{k}$ and add these to V_i and E_i to obtain V_{i+1} and E_{i+1} respectively.

Repeat step 2.)



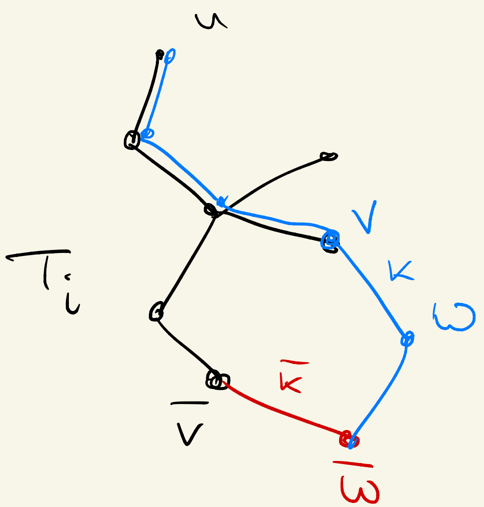
compare this with tree obtained from other choices when there was a tie.

Theorem 7.15 Dijkstra's alg. gives a spanning tree T with the property that the unique path from u to v is always a minimal u,v path in G with $d(u,v) = l(v)$ for all v.

<u>Proof</u> The alg. constructs a spanning tree.

Proof of minimality by induction. Let $T_i = (V_i, E_i)$. Then T_1 is the minimal path between u_0 and u_1. Suppose that the unique path in T_i from u_0 to any $u_j \in V_i$ is minimal in G. Suppose T_{i+1} is obtained by adding $\bar{k} = \bar{v}\bar{w}$. We must show that $l(\bar{w}) = l(\bar{v}) + w(\bar{k})$ is the weighted distance $d_w(u, \bar{w})$.

Let P be a shortest $u_0\bar{w}$ path in G and let v be the last vertex of P in T_i. Then



$$d(u_0,\bar{w}) = \ell(P_{u_0,v}) + w(k) + \ell(P(w,\bar{w}))$$

$$= \big(\ell(v) + w(k)\big) + \ell(P_{w,\bar{w}})$$

$$= \underbrace{f(k)}_{\geq \min f(k) = f(\bar{k})} + \underbrace{\ell(P(w,\bar{w}))}_{\geq 0}$$

$$\geq f(\bar{k}) = \ell(P_0)$$

Therefore, P_0 is a shortest path.

$\square$