Matchings and Networks

Matchings concern bipartite graphs.

<u>Definition 8.1</u> For a bipartite graph $G = (S+T, E)$ a matching $M \subseteq E$ is a set of mutually non-incident pair of edges

The <u>matching number</u> $m(G)$ of G is the # of edges in a maximally large matching

A matching is maximal if $M = m(G)$.

Ex. 1) $G = K_{m,n}$ $\quad m(G) = \min(n,m)$.

The number of max. matchings is

$n \le m$

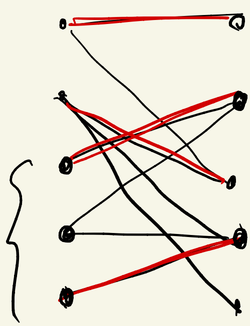
$$m(m-1)(m-2)\dots(m-n)$$

falling factorial.

A maximal matching corresponds to an injective mapping.

$f : A \to B \quad |A| = n \quad |B| = m$.

2)



$|M| = 4$ In fact

$m(G) = 4$

cannot match all $\underline{5}$!

3 only connect to 2 vertices.

Thm 8.3 Let $G = (S+T, E)$ be bipartite. Then $|S| = m(G)$ iff $|A| \le |N(A)|$ for all $A \subseteq S$ where

$$N(A) = \{ v \in T : uv \in E \text{ for some } u \in A \}$$

* Notice this is violated in the last example for $A = \{u_3, u_4, u_5\}$

Proof If $|A| > |N(A)|$ we cannot match all vertices in A, so $m(G) < |S|$.

For the converse, assume $|A| \le |N(A)|$ for all $A \subseteq S$. We will show

If M is a matching with $|M| < |S|$ then M is not maximal.

Let $u_0 \in S$ be a vertex not matched by M then since $|N(u_0)| \geq |\{u_0\}| = 1$ there exists a $v_1 \in T$ s.t. $u_0 v_1 \in E$. If v_1 is not matched by M, then M is not maximal since $M \cup \{u_0 v_1\}$ is a matching. So suppose v_1 is matched to $u_1 \neq u_0$ in M.

Since $|N(u_0, u_1)| \geq 2$ there is a vertex $v_2 \neq v_1$ that is a neighbor of u_0 or u_1. If v_2 is unmatched

Proceed to the next step.
Otherwise v_2 is matched to u_2 with $u_2 \notin \{u_0, u_1\}$ and there is a $v_3 \in N(u_0, u_1, u_2)$ with $v_3 \notin \{v_1, v_2\}$

Proceed in this way until finding unmatched $v_r \in N(\{u_0, \ldots, u_{r-1}\})$.

Then we can find a path

$$P = v_r u_a v_a u_b v_b \ldots\ldots v_h u_0$$

By assumption the edges

$$\underbrace{u_a v_a, u_b v_b \ldots\ldots, u_h v_h}_{k\text{-edges}} \quad \text{are in } M$$

and the edges

$$\underbrace{v_r u_a, v_a u_b, \ldots, v_h u_0}_{k+1 \text{ edges}} \quad \text{are not in } M.$$

Swapping out the k edges for the $k+1$ edges produces a matching M' with $|M'| = |M|+1$. $\square$

This can be strengthened!

Thm 8.4 Let $G = (S+T, E)$ be bipartite. Then

$$m(G) = \underbrace{|S| - \max_{A \subseteq S}(|A| - |N(A)|)}_{}$$

Proof Let δ = !! We know $m(G) \leq |S| - \delta$ and it suffices to prove equality. Consider $G' = (S + (T \cup D), E^*)$ a new graph obtained from G by adding to G all edges between S and D. For $A \subset S$ we have $N^*(A) = N(A) \cup D$ therefore $|N^*(A)| \geq |A|$. That is G^* has a matching M^* of order $|S|$ by Thm 8.3. Removing edges of M^* from S to D gives a matching of G of size $|S| - \delta$. So $m(G) = |S| - \delta$.

$\square$.

<u>Thm 8.5</u> For a bipartite graph $G = (S+T, E)$

$$\max\{|M| : M \text{ matching}\} = \min\{|D| : D \text{ a vertex cover}\}.$$