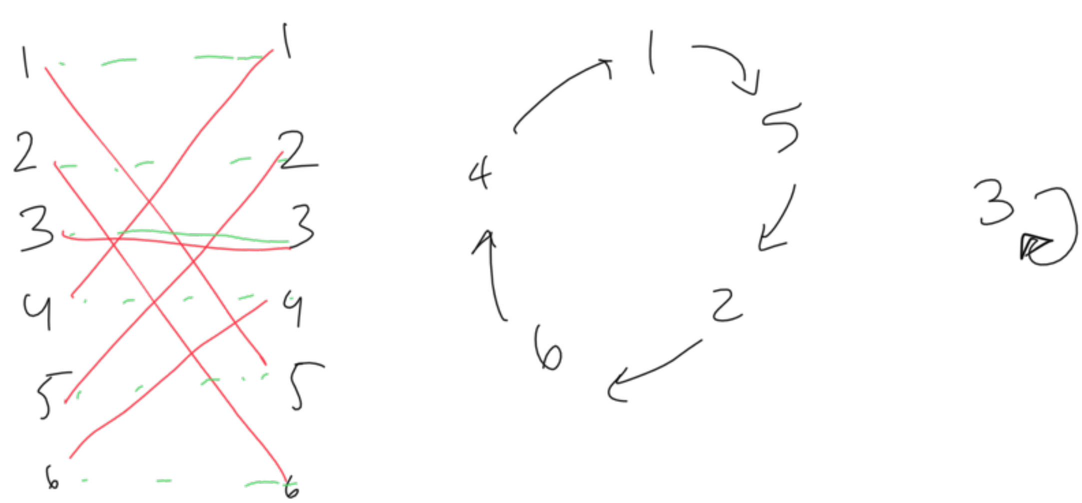
Permutation counting (1.3)

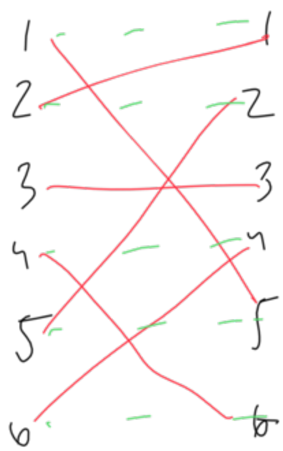
Permutation of N is a reordering of $\{1, \dots, n\} = N$

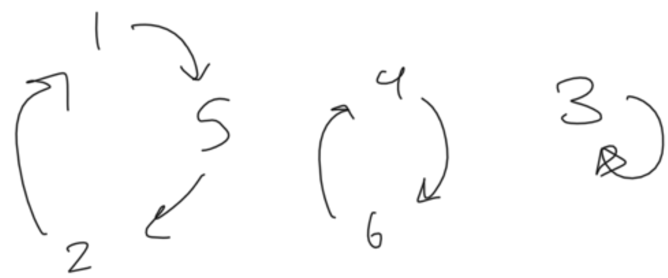
$\#$ permutations $= n!$

"symmetries of a finite set".



$$\pi_1 = (1,5,2,6,4)(3) = (3)(1,5,2,6,4)$$
$$= (3)(2,6,4,1,5)$$





$$\pi_2 = \underbrace{(3)(4,6),(1,2,5)}_{\text{can rearrange cycle order and rearrange inside cycles!!}}$$

$c_{n,k}$ = Stirling #'s of the first kind

= # of permutations of $\{1,\dots,n\}$ with k-cycles.

$$n! = \sum_{k=1}^{n} c_{n,k}$$

s.t. $n = \sum_{i=1} i\, b_i(\pi)$ $\quad b(\pi) = \sum b$

Def The type of a permutation is $(b_1^{(\pi)}, \ldots, b_n^{(\pi)})$ where

b_i = # cycles of length i

write:

$$t(\pi) = \underbrace{1^{b_1(\pi)} \cdots n^{b_n(\pi)}}_{\text{formal expression not to be evaluated!!}}$$

Proposition

$$\left\{ \begin{array}{l} \#\text{ of permutations} \\ \text{of type} \end{array} \right\} = \frac{n!}{1^{b_1} \cdots}$$

$$\{1^{b_1} \cdots\cdots n^{b_n}\} \qquad b_1! \cdots b_n! \, 1 \cdots n$$

Proof.

$$\underbrace{(\cdot)(\cdot)\ldots(\cdot)}_{b_1 \text{ 1-cycles}}\underbrace{(\cdot\;\cdot)(\cdot\;\cdot)\ldots(\cdot\;\cdot)}_{b_2 \text{ 2-cycles}}\ldots\ldots\underbrace{(\cdots\cdots)\cdots(\cdots\cdots)}_{b_n \text{ n-cycles}}$$

Fill in dots in $n!$ ways. Of course this is counting too much

1) Reorder b_i cycles among themselves in $b_i!$ ways. divide $n!$ by $b_1! \ldots b_n!$

2) For each cycle of length i we have i choices of where to start it. $\cdot b_i$

divide $\dfrac{n!}{b_1! \cdots b_n!}$ by L

for all i.

This proves formula $\square$.

<u>Proposition.</u>

$$n! = \sum_{(b_1 \ldots b_n)} \frac{n!}{b_1! \ldots b_n! \, 1^{b_1} \ldots n^{b_n}}$$

with $\sum i b_i = n$.

$$S_{n,k} = \sum_{(b_1, \ldots, b_n)} \frac{n!}{b_1! \ldots b_n! \, 1^{b_1} \ldots n^{b_n}}$$

with $\sum i b_i = n$ and

$$\sum_i b_i = k.$$

"Polynomial tricks" (Recurrence equations 1.4)

Return to binomial coefficients

For $r \in \mathbb{C}$ we can define

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\dots(r-k+1)}{k!} = \dfrac{r^{\underline{k}}}{k!} & k \geq \\ 0 & k < 0 \end{cases}$$

Proposition The recurrence

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}$$

holds for $r \in \mathbb{C}$.

Proof Polynomial method:
substitute indeterminate X for r.

$$f(X) = \binom{X}{K} \qquad g(X) = \binom{X-1}{K-1} + \binom{X-1}{K}$$

are both polynomials of degree K which agree at all positive integer points $X = n \in \mathbb{Z}_{\geq 0}$.

Since $\binom{n}{K} = \binom{n-1}{K-1} + \binom{n-1}{K}$.

Therefore the polynomials are equal and $\binom{r}{K} = \binom{r-1}{K-1} + \binom{r-1}{K}$

$\forall\ r \in \mathbb{C} \quad K \in \mathbb{Z}.$ $\square$

Proposition (Negation formula).

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}$$

$\forall r \in \mathbb{C},\ k \in \mathbb{Z}.$

Proof. $(-x)^{\underline{k}} = (-1)^k x(x+1)\dots(x+k-1).$

$$= (-1)^k x^{\overline{k}}$$

dividing by $k!$ on both sides and setting $r = x$ proves the equality $\square$.

(Newton's) Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof Via induction (not too enlightening).

Via the Taylor series for the function $f(x) = (x+a)^n$.

at $x = 0$.

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f}{dx^k}(0)\, x^k$$

$$\frac{d^k f}{dx^k}(0) = n(n-1)\dots(n-k+1)\, a^{n-k}.$$

For all $a \in \mathbb{C}$.

$$f(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k a^{n-k}.$$

Replacing a with y we obtain

$$\sum^{n} \left(n \right) \quad {}^{k} \quad {}^{n-k}$$

$$(x+y)^n = \sum_{k=0} \binom{n}{k} x \, y \qquad \square$$

<u>Proposition</u> (Vandermonde's identity).

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}.$$

<u>Proof</u>. Via polynomial argument.

We have.

$$(a+1)^{x+y} = (a+1)^x (a+1)^y.$$

Compare coefficient of a^n

$$\text{L.H.S} = \binom{x+y}{n} \qquad \text{R.H.S} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}.$$

$\square$.

proof Suffices to pro

Alternative proof ...

identity for $x=r$, $y=s$ natural #'s.

Let R, S be disjoint with

$|R| = r \quad |S| = s$.

$$\binom{r+s}{n} = \#\left\{\begin{matrix}\text{size } n \\ \text{subsets of} \\ R \cup S\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{size } n \\ \text{subsets} \\ \text{of } R \cup S\end{matrix}\right\} = \bigsqcup \underbrace{\left\{\begin{matrix} A \subset R \cup S \\ \text{s.t. } |A| = n \\ |A \cap R| = k \end{matrix}\right\}}_{\text{size } = \binom{r}{k}\binom{s}{n-k}}.$$

since if $|A \cap R| = k$ then $|A \cap S| = n-$

This proves the identity for all $r, s \in \mathbb{Z}_{\geq 0} \Rightarrow$ holds for

...

all $x, y \in \mathbb{C}$. $\square$.

Recurrences and polynomial equations for Stirling #'s.

Recall for Stirling #'s of the 2^{nd} kind

$$S_{n,k} = k\, S_{n-1,k} + S_{n-1,k-1}.$$

and $$r^n = \sum_{k=0}^{n} S_{n,k}\, r^{\underline{k}}.$$

$\Rightarrow \forall\ x \in \mathbb{C}.$

$$x^n = \sum_{k=0}^{n} S_{n,k}\, x^{\underline{k}}.$$

<u>Proposition</u> The Stirling #'s of

(1). 1st kind satisfy

The

$$\sigma_{n,k} = \sigma_{n-1,k-1} + (n-1)\,\sigma_{n-1,k}.$$

and

$$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \sigma_{n,k} x^k$$

<u>Proof</u> consider $i \in N$. A permutation either fixes i or not.

$$\left\{\begin{array}{l}\text{perms with} \\ k \text{ cycles} \\ \text{fixing } i\end{array}\right\} = \sigma_{n-1,k-1}$$

$$\left\{\begin{array}{l}\text{perms with} \\ k \text{ cycles} \\ \text{not fixing } i\end{array}\right\} = \underbrace{(n-1)\,\sigma_{n-1,k}}.$$

There are $n-1$ places to insert i into a permutation on $N \setminus i$. and obtain distinct permutations.

the

Taking the sum of the
two terms above yields

$$\sigma_{n,k} = \sigma_{n-1,k-1} + (n-1)\sigma_{n-1,k}.$$

For the polynomial equation
proceed by induction on n.
For $n=0$ it is trivial.

$$x^{\underline{n}} = x^{\underline{n-1}}(x-n+1)$$

$$= \left(\sum_{k=0}^{n-1} (-1)^{n-1-k} \sigma_{n-1,k} x^{k}\right)(x-n+$$

$$= \left(\sum_{k=0}^{n-1} (-1)^{n-1-k} \sigma_{n-1,k} x^{k+1}\right)$$

$$+ \sum_{k=0}^{n-1} (-1)^{n-k} (n-1)\sigma_{n-1,k} x^{k}$$

$$= \sum^{n} (-1)^{n-k} \left[\sigma_{n-1,k-1} + (n-1)\sigma_{n-1,k}\right]$$

$k=0$

$$= \sum_{k=0}^{n} (-1)^{n-1} s_{n,k}\, x^{k}$$

$\square$.

Some discrete probability

Question.

1) What is the expected # of fixed points of a permutation chosen at random?

2) What is the expected # of cycles of a perm chosen at random?

To "choose at random" we must

start from a probability space (Ω, p) Ω finite set $p: \Omega \to [0,1]$ s.t.

$$\sum_{\omega \in \Omega} p(\omega) = 1.$$

E.g. $\Omega = \{\text{permutations of } n\}$.

$$p: \Omega \to [0,1]$$
$$\pi \mapsto \frac{1}{n!}$$

$$|\Omega| = S \qquad p: \Omega \to [0,1]$$
$$\omega \mapsto \frac{1}{S}.$$

"Uniform distribution"

Def. A random variable is a mapping $X: \Omega \to \mathbb{R}$

eg. $X: \text{Perm}_n \to \mathbb{R}$

$$X(\pi) = \begin{cases} 1 & \pi \text{ has fixed point} \\ 0 & \text{otherwise} \end{cases}$$

or

$$X(\pi) = \#\text{fixed points.}$$

or

$$X(\pi) = \#\text{ cycles in } \pi.$$

The <u>expectation</u> of <u>X</u> is

$$EX := \sum p(\omega)\, X(\omega)$$

<u>Proposition</u> E is a linear function. i.e. if $X_1, \dots, X_m$ are random variables

$$E(\alpha_1 X_1 + \dots + \alpha_m X_m) = \sum_{i=1}^{m} \alpha_i E X_i$$

Answer to Q1.

The expected # of fixed pts of a random permutation on n elts is 1.

Let $F : \mathrm{Perm}_n \longrightarrow [0,1]$ be

$F(\pi) =$ # fixed pts of π

We want to calculate EF.

Let $F_i : \mathrm{Perm}_n \rightarrow [0,1]$ be

$$F_i(\pi) = \begin{cases} 1 & \text{if } i \text{ is fixed by } \pi \\ 0 & \text{otherwise} \end{cases}$$

# of permutations with i fixed is $(n-1)$

$$\Longrightarrow EF_i = \frac{1}{n!}(n-1)! = \frac{1}{n}.$$

Now $F = F_1 + \dots + F_n$

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So $E(F) = \sum E(F_i)$

$$E(F) = 1$$

Notice this doesn't depend on n !!!

Definition The variance is

$$VX = E\left((X - EX)^2\right).$$

measures the expected squared distance to the expected value

We have $VX = E(X^2) - EX^2$

The variance for F can also be calculated.

$$E(F^2) = E\left(\left(\sum_{i=1}^{n} F_i\right)^2\right)$$

$$= E\left(\sum_{i=1}^{n}\sum_{j=1}^{n} F_i F_j\right) \overset{\text{linearity}}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} E(F_i F_j)$$

$$= E(F_i^2) + 2E(F_i F_j).$$

$$E(F_i^2) = E(F_i) \quad \text{since } F_i = 0, 1.$$

$$E(F_i^2) = \frac{1}{n}.$$

$$E(F_i F_j) = \sum \{ p(\pi) : \pi \text{ has } i, j \text{ as fixed points} \}.$$

$$= \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$E(F^2) = 1 + 2\binom{n}{2}\left(\frac{1}{n(n-1)}\right).$$

$$= 2.$$

$$VF = E(F^2) - (EF)^2 = 2 - 1$$

$$= 1$$

-1.

Q2 What is the expected # of cycles of a permutation of N?

Let $X : \mathrm{Perm}_n \longrightarrow \mathbb{R}$

$\pi \longmapsto$ # of cycles.

$$EX = \sum_{\pi} \frac{1}{n!} X(\pi)$$

$$= \frac{1}{n!} \sum_{K=0}^{n} K \boxed{6_{n,K}}$$

— # perm with K cycles.

To evaluate this sum recall:

$$X^{\underline{n}} = \sum_{K=0}^{n} (-1)^{n-K} 6_{n,K} X^{K}$$

Differentiate by X:

Differentiate

$$\sum_{i=0}^{n-1} \prod_{\substack{j=0\\ i\neq j}}^{n-1} (x-j) = \sum_{k=0}^{n} (-1)^{n-k}\, k\, \sigma_{n,k}\, x^{k-1}$$

Substitute $x = -1$

$$\cancel{(-1)^{n-1}} \sum \prod_{\substack{i\neq j\\ j=0}}^{n-1} (j+1) = \cancel{(-1)^{n-1}} \sum_{k=0}^{n} k\, \sigma_{n,k}$$

Dividing by $n!$ on each side we obtain

$$\sum_{j=1}^{n} \frac{1}{j} = \frac{1}{n!} \sum_{k=0}^{n} k\, \sigma_{n,k} = EX.$$

$$\|$$

H_n n^{th} Harmonic #.