Ramsey Theory + Existence Theorems

The Pigeonhole principle

If n objects are distributed among $r < n$ containers, then there must exist at least one container with two objects

A well know corollary is:

If $f : N \to R$ is a map and $|N| > |R|$ then f

is <u>not</u> injective

Pigeonhole principle: Sharper version

$f: N \to R$ with $|N| = n > r = |R|$

There exists an $a \in R$ s.t.

$$|f^{-1}(a)| \geq \left\lfloor \frac{n-1}{r} \right\rfloor + 1$$

<u>Proof</u> by contradiction if

$|f^{-1}(a)| \leq \left\lfloor \frac{n-1}{r} \right\rfloor \quad \forall\ a \in R$ then

$$n = \sum |f^{-1}(a)| \leq r \left\lfloor \frac{n-1}{r} \right\rfloor < n.$$

Contradiction. $\square$

Example Let $a_1, \ldots, a_n$ be a collection of integers (not necess. distinct) Does there exist a subset whose sum is a multiple of n?

There are 2^n sums to consider (sums correspond to subsets.

$$N = \{0, a_1, a_1 + a_2, \ldots\ldots, a_1 + a_2 + \ldots + a_n\}$$

$$\downarrow f \quad \text{(take remainder when dividing by } n\text{)}$$

$$R = \{0, \ldots, n-1\}$$

By Pigeonhole principle there exist two sums with same remainder

ie. $$\sum_{i=1}^{\ell} a_i - \sum_{i=1}^{k} a_i = \sum_{i=k+1}^{\ell} a_i$$

is divisible by n.

Example 2 Let $a_1, --, a_{n^2+1} \in \mathbb{R}$ be distinct. Then there exists either $a_{k_1} < a_{k_2} < \dots < a_{k_{n+1}}$ $(k_1 < \dots < k_{n+1})$. or

$$a_{l_1} > \dots\dots > a_{l_{n+1}}$$

monotone subsequences.

For each a_i let $0 \leq t_i \leq n^2+1$ the length of a longest monotone increasing sequence beginning with a_i. Suppose $t_i \leq n$ (for a contradiction).

Then we have a mapping:

$$f: \{a_1, .., a_{n^2+1}\} \longrightarrow \{1, .., n\}$$

$a_i \longmapsto t_i$

By the PHP there exist $s \in \{1, \dots, n\}$ s.t. $|f^{-1}(s)| \geq \left\lfloor \frac{n^2}{n} \right\rfloor + 1 = n+1$

meaning there are $a_{l_1}, \dots, a_{l_{n+1}}$ (assume $l_1 < \dots < l_{n+1}$) s.t. the longest increasing monotone sequence starting from a_{l_i} is length $1 \leq s \leq n$. Consider $a_{l_i}, a_{l_{i+1}}$

If $a_{l_i} < a_{l_{i+1}}$ then there is an increasing seq. of length $s+1$ starting with a_{l_i} ie.

$$a_{l_i} < \underbrace{a_{l_{i+1}} < \dots\dots <}_{\text{sequence of length } s \text{ starting from } a_{l_{i+1}}}.$$

Therefore $a_{l_i} > a_{l_{i+1}}$ for all

i, and $a_{l_1} > \dots > a_{l_{n+1}}$.

$\square$

It is useful to think of a map $f: N \longrightarrow R$ as a "colouring" of a set N with $r = |R|$ colours.

Suppose $l_1, \dots, l_r$ are natural #'s and $n \geq l_1 + \dots + l_r - r + 1$

Then for every colouring of N ($|N| = n$) with r colours there must exist a colour i s.t. l_i elements are coloured i.

This is called the Ramsey property for $(l_1, \dots, l_r)$.

Suppose on the contrary that

Suppose ...

$|f^{-1}(i)| \leq l_i - 1 \quad \forall \; i$. Then

$$n = \sum |f^{-1}(i)| \leq \sum l_i - 1$$
$$n = l_1 + \dots + l_r - r$$

contradicting $n \geq l_1 + \dots + l_r - r + 1$

Consider the "party problem"

Let $k, l \geq 2$ natural #'s

What is the minimum # of guests you need to invite to a party so that there are k guests that all know each other OR l ...

guests that all don't know each other?

Example. If $l=2$ then for any k it suffices to invite k people. Either they all know each other or there is an unacquainted pair ($l=2$)

Thm Ramsey's Theorem

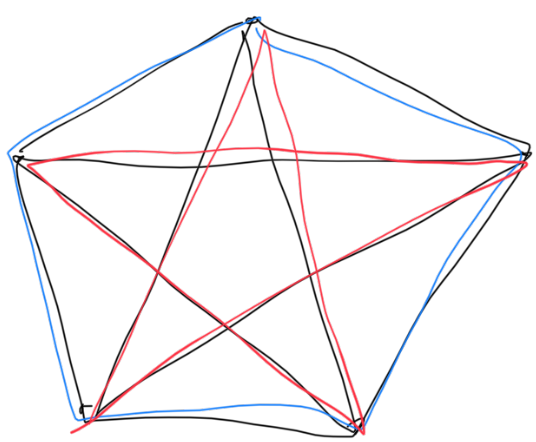
Let $k, l \geq 2$. Then there exists a least integer $\boxed{R(k,l)}$ (Ramsey #) s.t. if $|N| = n \geq R(k,l)$ and we colour all pairs from N either red or blue, then exists either a k-set there exists an

in N whose pairs are all coloured red or an l-set whose pairs are all colored blue.

Coloring pairs can be thought of in terms of graphs

let N be set of points and connect all points with edges

$|N| = 5.$



$R(k, 2) = k$ and

Proof Recall $R(k, l)$

$R(2, l) = l$.

We will prove $R(k, l)$ exists by induction. Namely it is bounded by:

$$R(k, l) \leq R(k-1, l) + R(k, l-1)$$

Suppose N satisfies

$|N| = n = R(k-1, l) + R(k, l-1)$.

and pairs of N are arbitrarily colored red + blue.

If $a \in N$ then $N \setminus a$ decomposes into $R \cup B$ where

$R = \{x \mid \{x, a\}$ is colored red$\}$

$B = \{x \mid \{x, a\}$ is colored blue$\}$

$$|R| + |B| = R(k-1, l) + R(k, l-1) - 1$$

Therefore either $|R| \geq R(k-1, \ell)$ ①

or $|B| \geq R(k, \ell-1)$ ②.

Assume ① then there is a $k-1$ set in R with all pairs colored red (together with a this gives a k-set), or there is an ℓ-set all colored blue.

Case ② follows analogously.

From the proof we also see that

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}$$

Example

Notice that $R(3,3) \leq 6$.

In fact $R(3,3) = 6$.

Suppose $R(3,3) = 5$ consider the colouring of the graph above. Are there any monochromatic triangles?

There are lower bounds on $R(m,n)$ using probabilistic methods (Erdös).

Exact values are unknown.

ie. $43 \leq R(5,5) \leq 48$ tighter.

$R(4,5) + R(4,5) = 50$.

"Dynamic survey" in Jour. of Elect. Comb.

$$798 \leq R(10,10) \leq 23556.$$

See page 32 - 33 for another coloring problem due to Erdös.