Counting derangements

Today's class will survey the counting techniques presented in Sections 2.1, 2.3, and 2.4 in the specific example of derangements. Note we are skipping section 2.2 (Calculus of finite differences).

Def A derangement is a permutation with no fixed points.

Let D_n be the number of

derangements in term n.

Claim $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

We will show this three different ways to highlight three different methods.

- Summation + solving recurrences
- Inversion formula 2.3
- Inclusion - exclusion 2.4

Summation If π is a derangement then $\pi(1) = i$ $i \neq 1$.

Two possibilities

- $\pi(i) = 1$ in two line

① $\pi(i) = 1$ in ...

notation

$$\pi = \begin{pmatrix} 1 & \cdots & i & \cdots & n \\ i & & 1 & & \pi(n) \end{pmatrix}.$$

Forgetting 1 and i gives a derangement of $n-2$ numbers. Therefore

$$\underbrace{(n-1)}_{\text{choice of } i} D_{n-2} \text{ derangements of this kind}$$

② $\pi(i) \neq 1$

replace by 1.

$$\pi = \begin{pmatrix} 1 & \cdots & i & \cdots & n \\ i & & \pi(i) & & \pi(n) \end{pmatrix}.$$

$$\pi(i) \neq 1$$

remove

$\Rightarrow$ we have a derangement of $n-1$ numbers. There are

$(n-1) D_{n-1}$ derangements of

$\underbrace{(n-1)}_{\text{choice of } i} D_{n-1}$ derangements of this kind

We obtain the recurrence

$$D_n = (n-1)\left(D_{n-1} + D_{n-2}\right)$$

notice $D_1 = 0 \quad D_2 = 1$

we can set $D_0 := 1$

Now manipulate

$$D_n - nD_{n-1} = -\left(D_{n-1} - (n-1)D_{n-\right.$$

$$= -\left((n-2)\left(D_{n-2} + D_{n-3}\right) - (n-1)D_{n-\right.$$

$$= D_{n-2} - (n-2)D_{n-3}$$

$$\vdots$$

$$= (-1)^{n-1}(D_1 - D_0) = (-1)$$

$$D_n = n D_{n-1} + (-1)^n.$$

$$D_0 = 1$$

Solving recurrences given by a summation formula of the form:

$$T_0 = \alpha$$

$$a_n T_n = b_n T_{n-1} + c_n$$

Simpler example: geometric series

$$S_n = 1 + q^1 + \cdots + q^n.$$

$$S_0 = 1$$

$$n+1$$

$$S_{n+1} = S_n + q^{n+1}$$

So $c_n = q^{n+1}$ $a_n, b_n = 1$.

$$S_{n+1} = 1 + q\sum_{k=0}^{n} q^k$$

$$S_n + q^{n+1} = 1 + q S_n$$

$$(1-q) S_n = 1 + q^{n+1}$$

$$\Rightarrow S_n = \frac{1 + q^{n+1}}{(1-q)}$$

Returning to T_n from above by multiplying recurrence relati by t_n where

$$t_n = \frac{t_{n-1} a_{n-1}}{1}$$

on

We get

$$t_n a_n T_n = t_{n-1} a_{n-1} T_{n-1} + t_n c_n$$

and set $S_n = t_n a_n T_n$.

$$S_n = S_{n-1} + t_n c_n$$

$$\Rightarrow S_n = \sum_{k=1}^{n} t_k c_k + t_0 a_0 T_0$$

$$T_n = \frac{\sum_{k=1}^{n} t_k c_k + t_0 a_0 T_0}{t_n a_n} \quad (*)$$

Returning to :

$$D_n = n D_{n-1} + (-1)^n$$

$D_n \quad nD_{n-1}$

we get $c_n = (-1)^n$.

$t_n = \dfrac{1}{n!}$, so $*$

yields

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Another approach <u>Inversion</u>.

Set $d(n,k) =$ # n-perms with exactly k fixed pts

Then $D_n = d(n,0)$

$$d(n,n) = 1$$

Also $d(n,k) = \underbrace{\binom{n}{k}}_{} D_{n-k}$

choice of who are
the k-fixed pts

Therefore

$$n! = \sum_{k=0}^{n} d(n,k) = \sum_{k=0}^{n} \binom{n}{k} D_{n-}$$

$$= \sum_{k=0}^{n} \binom{n}{k} D_k .$$

We have a sequence

$$V_n = \sum_{k=0}^{n} \binom{n}{k} u_n$$

want to switch it up to get a formula for u_n.

Think of this as a matrix eq'n

$$\underbrace{\begin{array}{r c}
 & \begin{pmatrix} \binom{0}{0} & 0 & \cdots & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & 0 & \cdots \\ & & \binom{i-1}{j-1} & & \\ \binom{n}{0} & \cdots & \cdots & \cdots & \binom{n}{n} \end{pmatrix} \\
 & j^{th} \text{ column}
\end{array}}_{= A} \begin{pmatrix} u_0 \\ \vdots \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_0 \\ \vdots \\ \vdots \\ v_n \end{pmatrix}$$

i^{th} row

If we know A^{-1} we are done

$$A^{-1} \begin{pmatrix} v_0 \\ \vdots \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_0 \\ \vdots \\ \vdots \\ u_n \end{pmatrix}.$$

What is a clever way to find A^{-1}?

Notice that by the binomial theorem we have the relation

$$(X-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} X^k$$

$$X^n = ((X-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (X-1)^k$$

Written in terms of matrices

$$\underbrace{\begin{pmatrix} \quad & & \quad \\ & & \\ & & \end{pmatrix}}_{=A} \begin{pmatrix} 1 \\ X-1 \\ (X-1)^2 \\ \vdots \\ (X-1)^n \end{pmatrix} = \begin{pmatrix} 1 \\ X \\ \vdots \\ X^n \end{pmatrix}$$

And

$$
\underbrace{\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
-\binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\
\binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & 0 \cdots & 0 \\
 & \vdots & & & \\
\pm\binom{n}{0} & \cdots & \cdots & & \binom{n}{n}
\end{pmatrix}}_{B}
\begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix}
=
\begin{pmatrix} 1 \\ x-1 \\ (x-1)^2 \\ \vdots \\ (x-1)^n \end{pmatrix}
$$

By applying the "polynomial method" 1.4 we obtain $B = A^{-1}$

Therefore we can "invert"

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_k$$

$$D_n = \sum (-1)^{n-k} \binom{n}{k} k!$$

$$= n! \sum \frac{(-1)^{n-k}}{\cdots} = n! \sum^{n} \frac{(-1)^k}{k!}$$

$\smile \; (n-k)! \qquad k=0$

Find the same formula again.

This phenomenon can be phrased generally

let $(p_0(x), p_1(x) \ldots, p_n(x) \ldots$

$(q_0(x), q_1(x) \ldots q_n(x) \ldots$

be sequences of polynomials wit

$\deg p_i(x) = \deg q_i(x) = i$.

Any polynomial of degree n can be expressed as a linear combination of $p_i(x)$'s (also q_i'

Even

$$q_n(x) = \sum a_{n,k} p_k(x)$$

$$p_n(x) = \sum b_{n,k}\, q_k(x)$$

This is linear algebra $(a_{n,k})$ $(b_{n,k})$ form matrices of $\mathbb{R}$ numb which are inverses of each other. Using this notation:

Thm Suppose $u_0, \ldots, u_n, \ldots$ $v_0, \ldots, v_n, \ldots$ are two sequences then

$$v_n = \sum_{k=0}^{n} a_{n,k} u_k \quad \forall n \iff u_n = \sum_{k=0}^{n} b_{n,k} v_k \quad \forall n$$

We just applied "binomial inversion" to obtain D_n.

Example: Consider $p_n(x) = x^n$

$q_n(x) = x^{\underline{n}} = x(x-1)\ldots(x-n+1)$.

Recall we proved

$$x^n = \sum S_{n,k} x^{\underline{k}} \quad \text{and}$$

$$x^{\underline{n}} = \sum (-1)^{n-k} \sigma_{n,k}$$

In particular $(S_{n,k})$ and $((-1)^{n-k}\sigma_{n,k})$ are inverse matrices. and

$$v_n = \sum_{k=0}^{n} S_{n,k} u_k \quad \forall n$$

$$\Longleftrightarrow \quad u_n = \sum_{k=0}^{n} (-1)^{n-k} \sigma_{n,k} v_k \quad \forall n$$

Lastly Inclusion - Exclusion

Lastly

Return to D_n derangements.

It is easier to

other i may be fixed here too.

$$D_n = n! - \sum_{i=1}^{n} \#\left\{\begin{array}{l}\text{perms with}\\ i \text{ fixed}\end{array}\right\}$$

$$+ \sum_{i \neq j} \#\left\{\begin{array}{l}\text{perms with}\\ i\ j \text{ fixed}\end{array}\right\}$$

$$- \sum_{i \neq j \neq k} \#\left\{\begin{array}{l}\text{perms with}\\ i, j, k \text{ fixed}\end{array}\right\}$$

$$+$$

$$\vdots$$

$$\pm \#\left\{\begin{array}{l}\text{perm with}\\ 1, \ldots, n \text{ fixed}\end{array}\right\}. = 1.$$

$$= n! - n! + \binom{n}{2}(n-2)!$$

$$- \binom{n}{3}(n-3)!$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Thm 2.3 Let $B_1, \dots, B_m$ be subset of a finite set S. Then

$$\left| S \setminus \bigcup_{i=1}^{m} B_i \right| = |S| - \sum_{i=1}^{m} |B_i|$$

$$+ \sum_{1 \le i < j \le m} |B_i \cap B_j| - \dots (-1)^m |B_1 \cap \dots \cap B_m|$$

Proof Count how many times $x \in S$ is counted on the RHS.

C $x \in B_{i_1} \dots B_{i_k}$ but not

Suppose

in the other B_j's.

Then x is counted

$$1 - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} \cdots\cdots (-1)^k \binom{k}{k}$$

This sum is 0 for $k \geq 1$.

1 if $k = 0$.
(in no B_i's).

Theorem 2.4 states the theorem above in terms of "properties".

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