Inclusion-Exclusion method.

Return to D_n derangements.

It is easier to

other i may be fixed here too.

$$D_n = n! - \sum_{i=1}^{n} \# \left\{ \begin{matrix} \text{perms with} \\ i \text{ fixed} \end{matrix} \right\}$$

$$+ \sum_{i \neq j} \# \left\{ \begin{matrix} \text{perms with} \\ i \ j \text{ fixed} \end{matrix} \right\}$$

$$- \sum_{i \neq j \neq k} \# \left\{ \begin{matrix} \text{perms with} \\ i, j, k \text{ fixed} \end{matrix} \right\}$$

$+$

$\vdots$

$\pm \#\{\text{perm with } 1, \dots, n \text{ fixed}\} = 1.$

$$= n! - n! + \binom{n}{2}(n-2)!$$

$$- \binom{n}{3}(n-3)!$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Thm 2.3 Let $B_1, \ldots, B_m$ be subsets of a finite set S. Then

$$\left|S \setminus \bigcup_{i=1}^{m} B_i\right| = |S| - \sum_{i=1}^{m} |B_i|$$

$$+ \sum_{1 \le i < j \le m} |B_i \cap B_j| - \cdots (-1)^m |B_1 \cap \cdots \cap B_m|$$

Proof Count how many times $x \in S$ is counted on the RHS.

Suppose $x \in B_{i_1}, \ldots, B_{i_k}$ but not in the other B_j's.

Then x is counted

$$1 - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} \cdots\cdots (-1)^k \binom{k}{k}$$

This sum is 0 for $k \geq 1$.

1 if $k = 0$.
(in no B_i's).

Theorem 2.4 states the theorem above in terms of "properties".

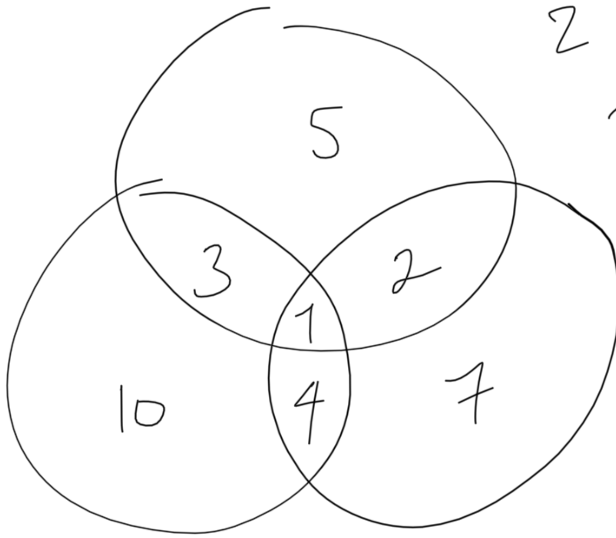
Example

There are 5 people who read fiction, 10 who read magazines, 7 who read non-fiction

3 who read both fiction + magazines

2 who read fiction & non-fiction

4 who read magazines and non-fiction.



How many readers are there in total?

$10 + 5 + 7 - 3 - 2 - 4 + 1$

$= 22 - 9 + 1 = 14$. Readers in total.

<u>Example</u> How many integers between 1 & 30 are relatively prime to 30?

$30 = 2 \cdot 3 \cdot 5$ prime factorisation.

$A_n = \# \left\{ \begin{matrix} \text{multiples of } n \text{ less than} \\ \text{or equal to } 30 \end{matrix} \right\}$

$|A_2| = 15$

$|A_3| = 10$

$|A_5| = 6$.

$S = S \setminus \underbrace{(A_2 \cup A_3 \cup A_5)}$.

$\{1, \ldots 30\}$

these sets overlap.
ie. $6 = 2 \cdot 3 \in A_2, A_3$.

$$|S \setminus A_2 \cup A_3 \cup A_5|$$
$$= |S| - |A_2| - |A_3| - |A_5| + |A_6| + |A_{10}| + |A_{15}|$$
$$- |A_{30}|$$
$$= 30 - 15 - 10 - 6 + 5 + 3 + 2 - 1.$$
$$= 8.$$

Example Partition counting

Set

$P_{odd}(n) =$ # of partitions of n with parts all odd

$P_{dist}(n) =$ # of partitions of n with parts all distinct.

$$P_{odd}(n) = P_{dist}(n)$$

$$P_{odd}(n) = P(n) - \sum |E_i| + \sum |E_i \cap E_j$$

$$- \sum |E_i \cap E_j \cap E_k| + \cdots$$

here E_i denotes the property that an even # i appears in the partiti

So we have:

$$P_{odd}(n) = P(n)$$

(one even part) $\quad - P(n-2) - P(n-4) - \cdots$

(two even parts) $\quad - P(n-2-4) - \cdots$

(three even parts) $\quad + P(n-2-4-6) + \cdots$

$\vdots$

Also let F_i be the property that

a partition into two parts of size i.

$$P_{dist}(n) = P(n) - \sum |E_i| + \sum |E_i \cap E_j|$$
$$- \sum |E_i \cap E_j \cap E_k|$$
$$+ \ldots$$

$$P_{dist}(n) = P(n) - \sum P(n-i-i)$$
$$+ \sum_{i<j} P(n-i-i-j-j)$$
$$- \sum_{i<j<k} P(n-i-i-j-j-k-k)$$
$$\vdots$$

We see that term by term we have equality

On the other hand it is not clear how to form a bijection between the two sets!!

Generating functions (Ch. 3)

Generating functions are a bookkeeping device for keeping track of infinite sequences. They also provide a connection between enumerative combinatorics and analysis with powerful implications.

<u>Def</u> The generating function

of (a_n) is the formal series

$$A(z) = \sum_{n \geq 0} a_n z^n$$

Here "formal" means we often don't care about convergence questions.

Generating functions can be manipul

Shifting indices:

$$zA(z) = \sum_{n \geq 1} a_{n-1} z^n$$

Summing sequences:

$$A(z) + B(z) = \sum (a_n + b_n) z^n$$

$A(z) + B(z) = \sum (a_n + b_n) =$

(additive identity is the series $a_n = 0$ $\forall n$.)

Convolution (Product of formal series)

$$A(z) \cdot B(z) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n$$

(multiplicative identity is $a_0 = 1$ $a_n = 0$ for $n > 0$.)

When does $A(z)$ posses an inverse? ie. $\exists$ $B(z)$ s.t.

$$A(z) \cdot B(z) = 1 \;?$$

$b_0 = a_0^{-1} \Rightarrow a_0 \neq 0$

When $a_0 \neq 0$ can solve step by step for b_n.

$$b_n = -a_0^{-1}\left(\sum_{k=1}^{n} a_k b_{n-k}\right).$$

Notice b_n are in $\mathbb{Q}$ if $a_0 \neq 1$.

Example. Inverse of $a_n = 1 \; \forall n$

$$\left(\sum_{n \geq 0} z^n\right) B(z) = 1.$$

$b_0 = 1$

$a_0 b_1 + a_1 b_0 = 0 \Rightarrow b_1 = -1.$

$b_2 + a_1 b_1 + a_2 b_0 = 0 \Rightarrow b_2 = \cdot$

$\vdots$

$b_n = 0 \qquad n > 1.$

$$\left(\sum z^n \right) (1 - z) = 1.$$

Inverses.

$$\sum_{n \geq 0} z^n = \frac{1}{1-z}$$ ⇐ "rational function

By substitution

$$\sum c^n z^n = \frac{1}{1 - cz}.$$

eg. $$\sum (-1)^n z^n = \frac{1}{1+z}.$$

We also have:

- $$\sum_{n\geq 0} \binom{c}{n} z^n = (1+z)^c \quad \forall c \in$$

(Notice this sequence terminates if $c \in \mathbb{N}$.)

- $$\sum_{n\geq 0} \binom{c+n-1}{n} z^n = \frac{1}{(1-z)^c}$$

(using negation formula for binomial coefficients)

Notice that $c=2$ we get:

$$\frac{1}{(1-z)^2} = \sum_{n\geq 0} (n+1) z^n \quad \text{and}$$

$$\frac{z}{(1-z)^2} = \sum_{n\geq 0} (n+1) z^{n+1}$$

Other series:

- $$\sum_{n\geq 0} \frac{z^n}{n!} = e^z$$

- $$\sum_{n\geq 1} \frac{(-1)^{n+1} z^n}{n} = \log(1+z)$$

These last two are well know Taylor series developmen

Solving recurrences using generati functions. (1200

Recall the Fibonacci #s.

$F_0 = 0$, $F_1 = 1$

$$F_n = F_{n-1} + F_{n-2}.$$

Let $F(z) = \sum_{n \geq 0} F_n z^n.$

Then the recurrence can be expressed as

$$F(z) = \underbrace{zF(z) + z^2 F(z)}_{\text{no constant term}} + \underset{F_1 z}{\underset{1}{z}} + (\quad F_{\cdot}$$

Solve for $F(z)$.

$$(1 - z - z^2) F(z) = z.$$

$$F(z) = \frac{z}{(1 - z - z^2)}.$$

How to find the coefficients of .

Recall partial fraction decompositio

$$\frac{1}{1-z-z^2} = \frac{1}{(1-\alpha z)(1-\beta z)} = \frac{a}{(1-\alpha z)} + \frac{b}{(1-\beta z)}$$

for some α, β, a, b.

Once these are found we have

$$F(z) = z\left(\frac{a}{1-\alpha z} + \frac{b}{1-\beta z}\right)$$

$$= z\left(a \sum \alpha^n z^n + b \sum \beta^n z^n\right)$$

$$= \sum \underbrace{\left(a\alpha^{n-1} + b\beta^{n-1}\right)}_{\overset{\shortparallel}{F_n}} z^n.$$

Quadratic formula

$$1 - z - z^2 = \left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right.$$

$$1 = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right). \quad \text{so,}$$

$$1 - z - z^2 = \left(1 - \frac{1-\sqrt{5}}{2} z\right)\left(1 - \frac{1+\sqrt{5}}{2} z\right.$$

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

To find a, b we considr

$$\frac{1}{1 - z - z^2} = \frac{a}{1 - \alpha z} + \frac{b}{1 - \beta z}.$$

and

$$1 = a - a\beta z + b - b\alpha z$$

so

$$a + b = 1$$

$$a\beta + b\alpha = 0$$

$$\Rightarrow \quad a = \frac{\alpha}{\sqrt{5}}, \quad b = \frac{-\beta}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

<u>Thm 3.1</u>. Let $q_1, \dots, q_d$ be complex #s $q_d \neq 0$ with

$$q(z) = 1 + q_1 z + \dots + q_d z^d = (1-\alpha_1 z)^{d_1} \cdots (1-\alpha_k z)^{d_k}$$

For a counting function (sequence) $f: \mathbb{Z}_{\geq 0} \to \mathbb{C}$ T.F.A.E.

1) f satisfies a recurrence

$$f(n+d) + q_1 f(n+d-1) + \dots + q_d (f_n) =$$

for all $n \geq 0$.

2) The generating function for f can be written as a ratio function

$$F(z) = \sum_{n \geq 0} f(n) z^n = \frac{p(z)}{q(z)}$$

where $p(z)$ is a polynomial of degree $< d$.

3) There is a partial fraction decomposition

$$F(z) = \sum f(n) z^n = \sum \frac{g_i(z)}{(1-\alpha_i z)^{d_i}}$$

for polynomials $g_i(z)$ of degree less than d_i.

4) Explicit representation

$$f(n) = \sum_{i=1}^{k} p_i(n) \alpha_i^n$$

$p_i(n)$ are polynomials of degree less than d_i.

<u>Example</u>. Let B_n be the # of words formed with a, b, c s.t. "aa" does not appear.

$B_0 := 1.$

$B_1 = 3$ a b c.

$B_2 = 8$ ab ac ba bb b
ca cb cc

$\vdots$

$$B_n = \underbrace{2B_{n-1}}_{\text{words with first letter b or c.}} + \underbrace{2B_{n-2}}_{\text{words with first letter a second letter b or c}}$$

The relation in terms of generating functions is then

$$B(z) = 2zB(z) + 2z^2B(z) + 3z +$$

1 + 3z

$$B(z) = \frac{\quad}{(1 - 2z - 2z^2)}.$$

$$\vdots$$