Solving recurrences with generating functions.

Recall the Fibonacci #'s

$F_0 = 0 \ , F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$

$1, 1, 2, 3, 5, 8, 13, \ldots$

$$F_n = F_{n-1} + F_{n-2}.$$

Let $F(z) = \sum_{n \geq 0} F_n z^n.$

Then the recurrence can be expressed as

$$F(z) = \underbrace{zF(z) + z^2 F(z)}_{\text{no constant term}} + \underset{F_1 z}{\underset{1}{z}} + (\quad F$$

Solve for $F(z)$.

$$(1 - z - z^2) F(z) = z.$$

$$F(z) = \frac{z}{(1 - z - z^2)}.$$

How to find the coefficients of .

Recall partial fraction decompositio

$$\frac{1}{1-z-z^2} = \frac{1}{(1-\alpha z)(1-\beta z)} = \frac{a}{(1-\alpha z)} + \frac{b}{(1-\beta z)}$$

for some α, β, a, b.

Once these are found we have

$$F(z) = z\left(\frac{a}{1-\alpha z} + \frac{b}{1-\beta z}\right)$$

$$= z\left(a\sum \alpha^n z^n + b\sum \beta^n z^n\right)$$

$$= \sum \underbrace{\left(a\alpha^{n-1} + b\beta^{n-1}\right)}_{= F_n} z^n.$$

Quadratic formula

$$1 - z - z^2 = \left(z - \frac{1+\sqrt{5}}{2} \right)\left(z - \frac{1-\sqrt{5}}{2} \right.$$

$$1 = \left(\frac{1+\sqrt{5}}{2} \right)\left(\frac{1-\sqrt{5}}{2} \right). \quad \text{so,}$$

$$1 - z - z^2 = \left(1 - \frac{1-\sqrt{5}}{2} z \right)\left(1 - \frac{1+\sqrt{5}}{2} z \right.$$

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

To find a, b we considr

$$\frac{1}{1 - z - z^2} = \frac{a}{1 - \alpha z} + \frac{b}{1 - \beta z}.$$

and

$$1 = a - a\beta z + b - b\alpha z$$

so

$$a + b = 1$$
$$a\beta + b\alpha = 0$$

$$\Rightarrow \quad a = \frac{\alpha}{\sqrt{5}}, \quad b = \frac{-\beta}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

<u>Thm 3.1</u>. Let $q_1, \ldots, q_d$ be complex #s $q_d \neq 0$ with

$$q(z) = 1 + q_1 z + \ldots + q_d z^d$$
$$= (1-\alpha_1 z)^{d_1} \cdots (1-\alpha_k z)^{d_k}$$

For a counting function (sequence) $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ T.F.A.E.

1) f satisfies a recurrence

$$f(n+d) + q_1 f(n+d-1) + \dots + q_d (f n) =$$

for all $n \geq 0$.

2) The generating function for f can be written as a ratio function

$$F(z) = \sum_{n \geq 0} f(n) z^n = \frac{p(z)}{q(z)}$$

where $p(z)$ is a polynomial of degree $< d$.

3) There is a partial fraction decomposition

$$F(z) = \sum f(n) z^n = \sum \frac{g_i(z)}{(1-\alpha_i z)^{d_i}}$$

for polynomials $g_i(z)$ of degree less than d_i.

4) Explicit representation

$$f(n) = \sum_{i=1}^{k} p_i(n) \alpha_i^n$$

$p_i(n)$ are polynomials of degree less than d_i.

<u>Example</u>. Let B_n be the # of words formed with a, b, c s.t. "aa" does not appear.

$B_0 := 1.$

$B_1 = 3$ a b c.

$B_2 = 8$ ab ac ba bb b
ca cb cc

$\vdots$

$$B_n = \underbrace{2B_{n-1}}_{\text{words with first letter b or c.}} + \underbrace{2B_{n-2}}_{\text{words with first letter a second letter b or c}}$$

The relation in terms of generating functions is then

$$B(z) = 2zB(z) + 2z^2B(z) + 3z +$$

$1 + 3z$

$$b(z) = \frac{}{(1 - 2z - 2z^2)}.$$

$\vdots$