Generating functions continued.

Example Find the generating function and a closed formula for # of partitions of n with parts in $\{1, 2\}$.

From last time:

$$F(z) = (1+z+z^2+\dots)(1+z^2+z^4+\dots)$$

$$= \frac{1}{1-z} \cdot \frac{1}{1-z^2}$$

Alternatively find a recurrence. Let f_n denote the # of such partitions of n. Then

$$f_n = f_{n-2} + \underbrace{1}$$ — partition with only ones.

ie. $$F(z) = z^2 F(z) + (1+z+z^2+\dots).$$

$$\Rightarrow (1-z^2) F(z) = \frac{1}{1-z}$$

$$F(z) = \frac{1}{(1-z)} \cdot \frac{1}{(1-z^2)}$$

$$= \frac{1}{(1-z)^2} \cdot \frac{1}{1+z}$$

Now find partial fraction decomposition. Notice $d_1 = 2$, so

$$F(z) = \frac{1}{(1-z)^2} \cdot \frac{1}{1+z} = \frac{g_1(z)}{(1-z)^2} + \frac{g_2(z)}{1+z}$$

where $g_1(z)$ is of degree < 2
and $g_2(z)$ is of degree < 1.

Solve:

$$\frac{1}{(1-z)^2(1+z)} = \frac{az+b}{(1-z)^2} + \frac{c}{1+z}$$

$$\Rightarrow 1 = az + az^2 + b + zb + c - 2cz + cz^2$$

$b + c = 1$

$a + b - 2c = 0$ $\Rightarrow$

$a + c = 0$

$-c = a$

$b = 3c$

$c = 1/4$.

$a = -1/4$

$b = 3/4$.

$$F(z) = \frac{1}{4} \cdot \left[[3 - z] \sum_{n=0}^{\infty} \binom{n+1}{n} z^n + (-1)^n z^n \right]$$

$$F(z) = \frac{1}{4}\left[\sum_{n=0}^{\infty}\left[3(n+1)-n+(-1)^n\right]z^n\right]$$

$$= \frac{1}{4}\left[\sum_{n=0}^{\infty} 2n+3+(-1)^n\right].$$

$$f_n = \begin{cases} \frac{1}{4}\cdot(2n+4) & n \text{ even} \\ \frac{1}{4}\cdot(2n+2) & n \text{ odd} \end{cases}$$

$$f_n = \begin{cases} \frac{n}{2}+1 & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd.} \end{cases}$$

Extra topic Catalan #'s.

One of the most ubiquitous sequences in combinatorics.

Consider # of ways of arranging n pairs of brackets.

$C_0 := 1$ $\qquad \emptyset$

$C_1 = 1$ $\qquad$ ()

$C_2 = 2$ $\qquad$ (()) $\quad$ ()()

$C_3 =$ $\qquad$ ()(()) $\quad$ ()()() $\quad$ (())() $\quad$ ((())) $\quad$ (()())

$\vdots$ $\qquad$ $C_0C_2 \qquad C_1C_1 \qquad C_2C_0$

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

Notice that

$$\sum_{i=0}^{n} C_i C_{n-i}$$ is the n^{th} coefficient of $C(z)^2$ where

$$C(z) = \sum_{n \geq 0} C_n z^n.$$

Therefore,

$$C(z) = \underbrace{z\,C(z)^2}_{\text{shift by } z.} + \underbrace{1}_{\text{constant term } C_0.}$$

We can solve this as if it is a quadratic equation in $C(z)$!

$$z\,C(z)^2 - C(z) + 1 = 0$$

$$\Rightarrow \quad C(z) = \frac{1 \pm \sqrt{1-4z}}{2z}$$

Which function do we choose "$\pm$"?

Recall that:

$$\sqrt{1-4z} = (1-4z)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4)^n z^n$$

$$= 1 - 4\left(\tfrac{1}{2}\right)z + \cdots\cdots$$

$C(z)$ has no terms with negative exponents in z. So we must take $C(z) = \dfrac{1-\sqrt{1-4z}}{2z}$

A closed formula for C_n

$$C_n = -\frac{(-4)^{n+1}}{2}\binom{\frac{1}{2}}{n+1}$$

$$= (-1)^n \, 2^{2n+1}\binom{\frac{1}{2}}{n+1}$$

$$= (-1)^n \, 2^{2n+1} \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{(2n-1)}{2}\right)}{(n+1)!}$$

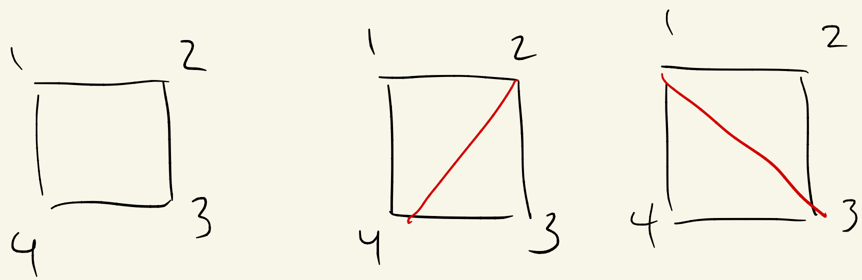
$$= \frac{2^{2n+1} \, 2^{-(n+1)}}{(n+1)!} \, 1\cdot 3\cdots(2n-1)$$

$$= \frac{\cancel{2^n}}{(n+1)!}\cdot\frac{(2n)!}{\cancel{2^n}\, n!} = \frac{1}{n+1}\binom{2n}{n}$$

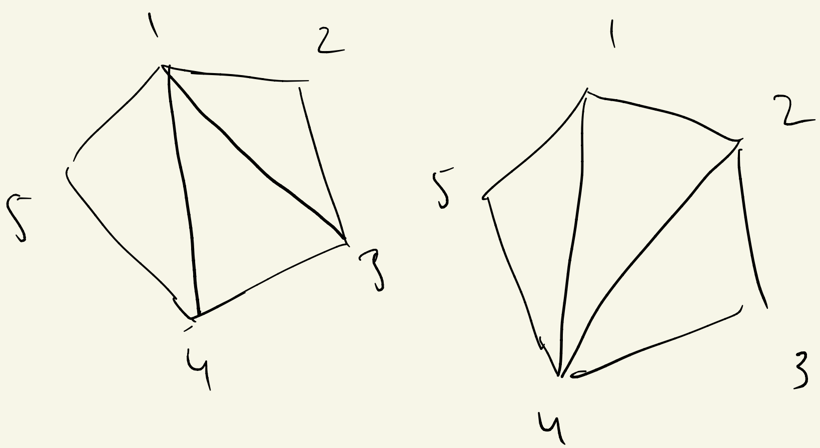
The Catalan numbers enumerate many other objects

<u>Exercise</u> Show the # of ways of triangulating a <u>labelled</u> $n+2$-gon is C_n

<u>eg.</u>



$C_2 = 2$.



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$C_3 = 5$

Generating functions of exponential type (3.3)

Another practical book keeping device is the exponential generating function of a sequence (a_n)

$$\hat{A}(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$$

we can add, multiply and convolute these series

$\hat{C}(z) = \hat{A}(z)\hat{B}(z)$ then

$$\frac{c_n}{n!} = \sum \frac{a_k b_{n-k}}{k!(n-k)!} \qquad c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

Example

$$e^{az} = \sum \frac{a^n}{n!} z^n$$

$$e^{bz} = \sum \frac{b^n}{n!} z^n$$

$$e^{az} e^{bz} = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\parallel$$

$$e^{(a+b)z} = \sum \frac{(a+b)^n}{n!} z^n.$$

We obtain the binomial theorem

$$(a+b)^n = \sum \binom{n}{k} a^k b^{n-k}$$

immediately from the exponential g.f.

Consider again the number D_n of derangements on $\{1, \dots, n\}$

Recall we found

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_k$$

So, $$\hat{D}(z) = \sum \frac{D_n}{n!} z^n$$

$$e^z = \sum \frac{1}{n!} z^n$$

give:

$$\hat{D}(z) e^z = \sum \frac{n!}{n!} z^n = \frac{1}{1-z}$$

$$\hat{D}(z) = \frac{e^{-z}}{1-z}$$ ← the n^{th} coefficient is the sum of the first n terms of e^{-z}

ie. $$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

This is a generating function formulati[on] of "binomial inversion"

$$\hat{V}(z) = \hat{U}(z) e^{z} \iff e^{-z} \hat{V}(z) = \hat{U}(z)$$

$$\hat{U}(z) = \sum \frac{u_n}{n!} z^n$$

$$\hat{V}(z) = \sum \frac{v_n}{n!} z^n$$