Chapter 5 Asymptotic Analysis

This technique can be used to understand the growth of sequences even when exact formulas cannot be found.

Def Let $f(n), g(n)$ be sequences

We say $g(n)$ grows faster than $f(n)$ $\quad f(n) \prec g(n)$ if $\forall\, \varepsilon > 0$ there exists $n_0(\varepsilon)$ s.t.

$$|f(n)| \leq \varepsilon\, |g(n)| \quad \forall\, n \geq n_0(\varepsilon)$$

- If $f(n) \prec g(n)$ and $g(n) \equiv 0$ then $f(n) = 0$ for $n \geq n_0(\varepsilon)$ for some $n_0(\varepsilon)$.

- If $g(n)$ has only finitely many zeros then

$$f(n) \prec g(n) \iff \lim_{n\to\infty} \left| \frac{f(n)}{g(n)} \right| = 0.$$

- The relation $\prec$ is transitive

ie. $f(n) \prec g(n)$, $g(n) \prec h(n)$ then $f(n) \prec h(n)$.

- $n^a \prec n^b$ for $a, b \in \mathbb{R}$ $a \leq b$

- We have the hierarchy:

$$c \prec \log n \prec n^{\varepsilon} \prec c^{n} \prec n^{n}$$

Intuitive from calculus

$$\log\log n \prec \log n \quad \text{and so on.}$$

Where is $n!$ in this hierarchy?

$$n! \prec n^{n} \qquad \text{since } k! < k^{k}, \; k > 1$$

yet $c^{n} \leq n!$ for large enough n (depending on c)

so $c^{n} \prec n!$

Definition Big O notation

$$O(g(n)) = \left\{ f(n) \;\middle|\; \begin{array}{l} \exists \text{ a constant } C>0 \text{ s.t.} \\ |f(n)| \leq C|g(n)| \\ \text{for all } n \geq n_0 \; \exists n_0 \end{array} \right\}$$

Notice $O(g(n))$ is a <u>set</u>.

Nonetheless it is convention to write $f(n) = O(g(n))$ when we really mean $f(n) \in O(g(n))$.

<u>Example</u> $p(n) = 2n^3 - n^2 + 6n + 100$

$p(n) = O(n^3)$ since

$$|p(n)| \leq 2|n^3| + |n^2| + 6|n| + 100$$

$$\leq 2|n^3| + |n^3| + 6|n^3| + 100 \leq 10|n^3|$$

for $n \geq 5$

Notice we could have also written $p(n) = O(n^4), O(n^5)$.
these are <u>not</u> the <u>best</u> possible estimates.

<u>Def</u> For estimates from below consider

$$\Omega(g(n)) = \left\{ f(n) \;\middle|\; \begin{array}{l} \exists \text{ a constant } C > 0 \text{ s.t.} \\ |f(n)| \geq C|g(n)| \text{ for } n \geq n_0 \end{array} \right.$$

again write $f(n) = \Omega(g(n))$ if $f(n) \in \Omega(g(n))$.

Notice $f(n) = O(g(n)) \iff$
$g(n) = \Omega(f(n))$

Both O and Ω are transitive.

To incorporate both:

$$\Theta(g(n)) = \left\{ f(n) \;\middle|\; \begin{array}{l} \exists \text{ constants } c_1, c_2 > 0 \\ \text{s.t.} \\ c_1 |g(n)| \le |f(n)| \\ \qquad \le c_2 |g(n)| \\ \text{for } n \geq n_0. \end{array} \right\}$$

We can also write

$$f(n) \asymp g(n) \iff f(n) = \Theta(g(n))$$
$$\iff g(n) = \Theta(f(n))$$

Even stronger version of this is:

$$f(n) \sim g(n) \iff \lim_{n\to\infty} \left| \frac{f(n)}{g(n)} \right| = 1$$

Def If $f(n) \sim g(n)$ we say $f(n)$ and $g(n)$ are asymptotically equal.

Example Polynomials provide examples of all three notions

1) $p(n) \prec q(n) \iff \deg p(n) < \deg q(n)$

2) $p(n) \asymp q(n) \iff \deg p(n) = \deg q(n)$

3) $p(n) \sim q(n) \iff$ $p(n) \asymp q(n)$ and leading coefficients are equal in absolute value.

Example for fixed $k>0$ what is the asymptotic of $\binom{n}{k}$?

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

$$= \frac{n^k - (1+2+\dots+(k-1))n^{k-1} + \dots + (-1)(-2)\dots(-k+1)n}{k!}$$

Therefore it is polynomial in n of degree k

For fixed k

$$\binom{n}{k} \sim \frac{1}{k!} n^k.$$

What about $n! = n(n-1)\ldots(2)(1)$.

This is not polynomial in n of fixed degree $\underline{\underline{d}}$.

Stirling's formula:

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \sqrt{2\pi}.$$

so that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

This can be expressed and proved logarithmically

$$\log n! = n\log n - n + \frac{\log n}{2} + \log\sqrt{2\pi} + R(n)$$

where $R(n) \to 0$ as $n \to \infty$.

Some other know asymptotics

Stirling #'s of second kind

For K fixed $n \to \infty$

$$S_{n,K} \sim \frac{K^n}{K!}$$

$S_{n,K}$ = # of set partitions of $\{1, \ldots, n\}$ into K- subsets.

Stirling #'s of the first kind

$\sigma_{n,K}$ = # of perms on $\{1, \ldots, n\}$ with K cycle

For n fixed as $K \to \infty$.

$$\sigma_{n+K,K} \sim \frac{K^{2n}}{2^n n!}$$

Catalan #'s recall

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}.$$

$$C_n \sim \frac{4^n}{\sqrt{\pi}\, n^{3/2}}\left(1 + O(1/n)\right).$$

Partitions

$$P_n \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{\frac{2n}{3}}}$$

$$\log P_n \sim \pi\sqrt{\frac{2n}{3}}$$

$$P_n^{dist} \sim \frac{3^{3/4}}{12 n^{3/4}}\, e^{\pi\sqrt{\frac{n}{3}}}$$

$\Rightarrow \log n! \sim n\log n - n + \frac{\log n}{2}$

Last bit of notation "little o notation"

$o(g(n)) = \{ f(n) :$ $\forall\ \varepsilon > 0$ there exists $n_0(\varepsilon)$ s.t. $|f(n)| \leq \varepsilon |g(n)|$ for $n \geq n_0(\varepsilon)$ $\}$

By convention we write:

$f(n) = o(g(n))$ to mean

$f(n) \in o(g(n))$.

i.e. $f(n) = o(1) \Rightarrow f(n) \to 0$

while $f(n) = O(1)$ means

$f(n)$ is bounded above by a constant C.

Notice that little o is a stronger statement than big O.

$$f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n)).$$

converse is <u>not</u> <u>true</u>.

$$x^2 \notin o(x^2) \qquad x^2 \in O(x^2).$$

$$x^2 \in o(x^3).$$

Order of magnitude for recurrence relations:

Recall Fibonacci sequence: we found

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right)$$

Also $|\hat{\phi}_n| < 1$ therefore

$$F_n \sim \frac{1}{\sqrt{5}} \phi^n \qquad \phi = \frac{1+\sqrt{5}}{2}$$

less precisely $F_n \asymp \phi^n$.

changing initial conditions wouldn't change anything

The generating function would be $\tilde{F}(z) = \dfrac{c+dz}{1-z-z^2}$

and we would obtain

$$\tilde{F}_n = a\phi^n + b\hat{\phi}^n.$$

For recurrences with fixed length and constant coefficients (Thm 3.1) we can use explicit formulas to find asymptotics.

What about other recurrences?

<u>Example</u>. ~~Tournament~~. Suppose $2^k = n$ players are in a tournament with losers eliminated. How many rounds must be played?

$$T(n) = T(n/2) + 1, \quad T(1) = 0.$$

$$T(2^k) = T(2^{k-1}) + 1$$

$$= T(2^{k-2}) + 2 \ldots$$

$$= T(1) + k = k.$$

$T(n) = \log_2(n) = k$ for $n = 2^k$.

What about when $n \neq 2^k$.

Then $\lfloor \frac{n}{2} \rfloor$ matches are played in first round and players are eliminated so.

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1 \quad T(1) = 0.$$

$$T(n) = \lceil \log_2 n \rceil$$

In general we can consider asymptoticals

$$T(n) = aT(n/b) + f(n)$$

$$T(1) = c$$

where $\frac{n}{b}$ means either $\lceil \frac{n}{b} \rceil$ or $\lfloor \frac{n}{b} \rfloor$.

Thm 5.2 $a \geq 1, b \geq 1$ $T(n) = aT(\frac{n}{b}) + f(n)$

a) If $f(n) = O(n^{\log_b a - \varepsilon})$ $\exists \varepsilon > 0$ then $T(n) = \Theta(n^{\log_b a})$

b) If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a})$

c) If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$ and $af(\frac{n}{b}) \leq c f(n)$ for some $c < 1$ and $n \geq n_0$ then $T(n) = \Theta(f(n))$.

See text for the proof.