Graph Theory

A graph $G = (V, E)$ consists of a finite set V - vertices and a set $E \subseteq \binom{V}{2}$ of pairs $\{u, v\}$ $(u \neq v)$ called Edges

Usually represent graphs pictorially.

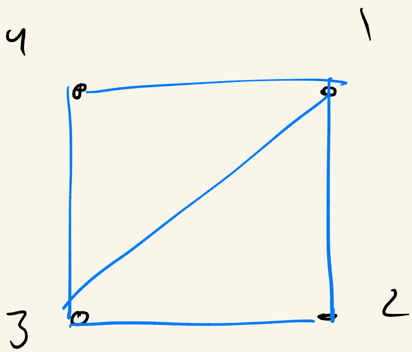
V vertices correspond to points

E edges correspond to segments joining points.

Eg.

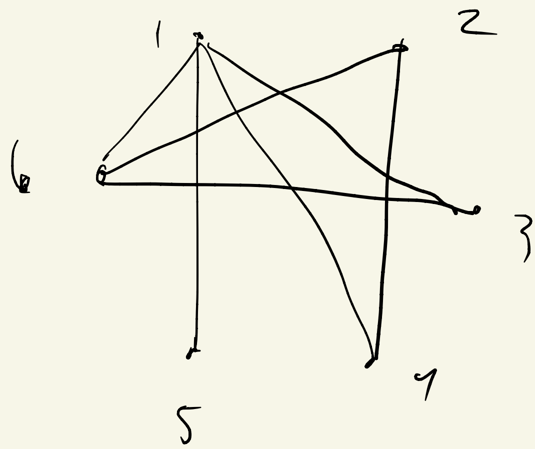
$$V = \{1, 2, 3, 4\}$$

$$E = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}\}.$$



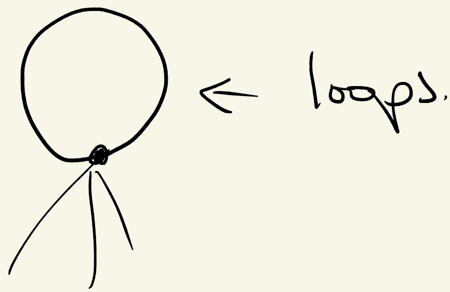
Any set with a binary relation (symmetric) can be considered as a graph: Consider $\{1, \ldots, n\}$ with relation $\{i,j\} \iff i \mid j$ or $j \mid i$.

Then for $n = 6$.



A number is prime $\iff$ there is only one edge eminating.

Notice our graph definition does not allow multiple edges (E is a set not a multiset). nor does it allow loops

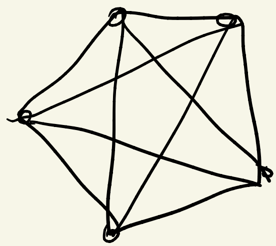


Some common / important graphs.

1) The complete graph on n vertices K_n.

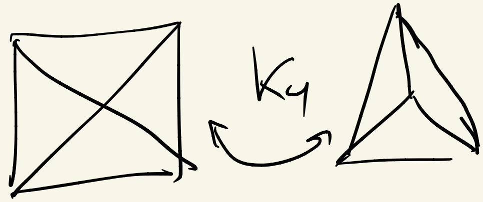
$V = \{1, \dots, n\}$

$E = \binom{V}{2}$.



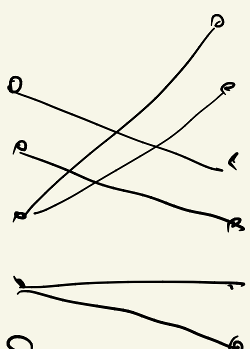
K_5.

$|E| = \binom{n}{2}$.



K_4

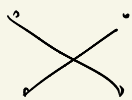
2) A graph is **bipartite** if $V = S \sqcup T$ and $\forall \{u,v\} \in E$ $u \in S$, $v \in T$ or vice versa.



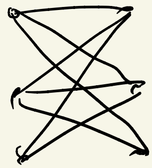
S T.

edges only between S and T

3. The complete bipartite graph $K_{n,m}$ contains an edge for each pair $u \in S$ $v \in T$.



$K_{2,2}$

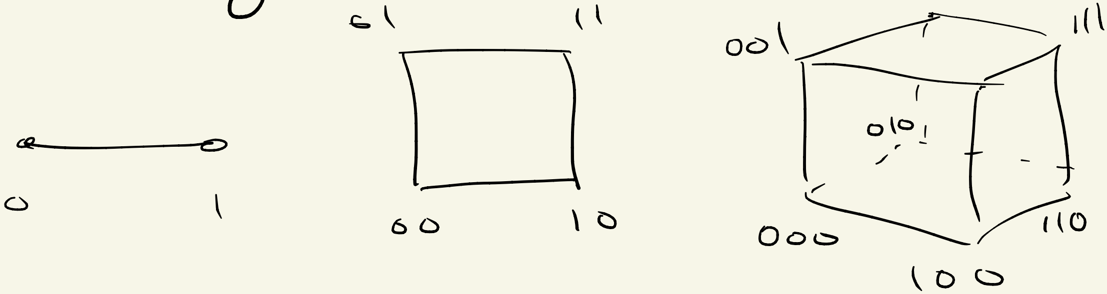


$K_{3,3}$.

Bipartite graphs are useful in assignment problems person-task etc.

4. A hypercube Q_n has 2^n vertices corresponding to binary strings of length n. The Q_n has an edge between two vertices if the strings differ

in exactly one place.



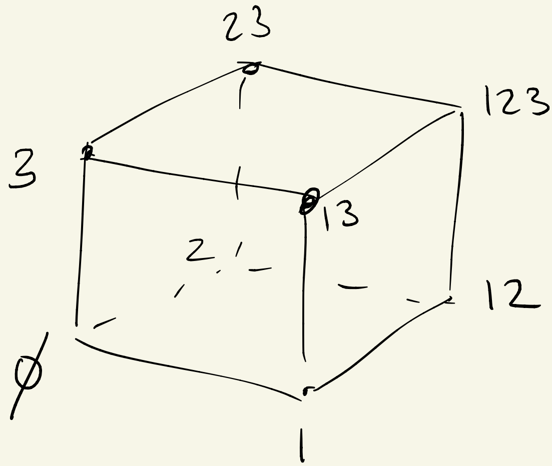
Vertices in Q_n will be useful in coding theory.

Exercise.
How many edges are in Q_n?

Hint: use induction noting that Q_n splits into two copies of Q_{n-1} joined by some edges.

Recall that 0,1 words of length n are in bijection with the set of subsets of $\{1, \ldots, n\}$

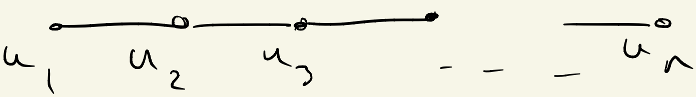
Therefore we can also view vertices of Q_n as subsets of $\{1, 2, 3\}$ with an edge between A and B iff



$A = B \sqcup i$ or $B = A \sqcup i$ for some i.

"Boolean lattice"

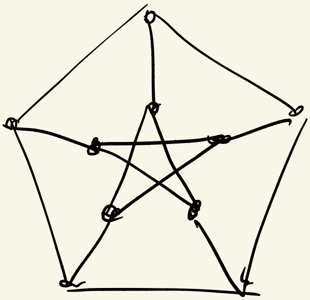
5. A *path* is a graph P_n with n vertices that can be ordered so that $\{u_i, u_{i+1}\}$ are the only edges.



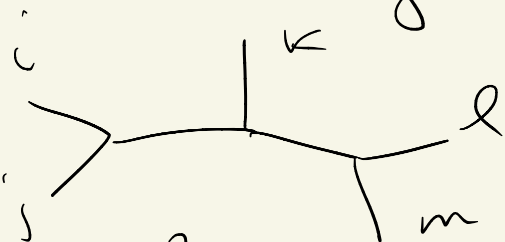
A *circuit* C_n is a path P_n with an additional edge $\{u_1, u_n\}$

6. The Peterson graph.

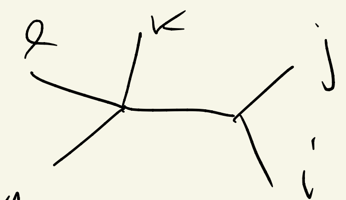
10 vertices
15 edges.



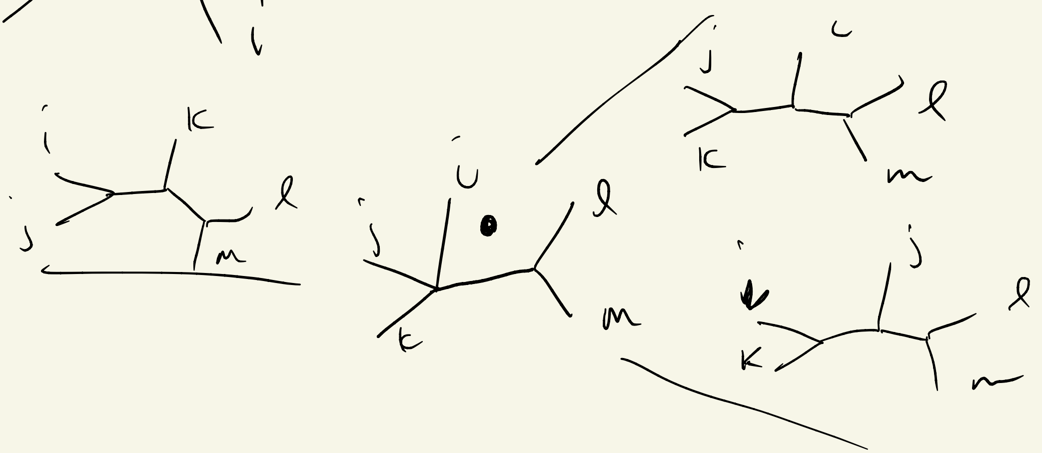
<u>Exerise</u> label the edges with trees



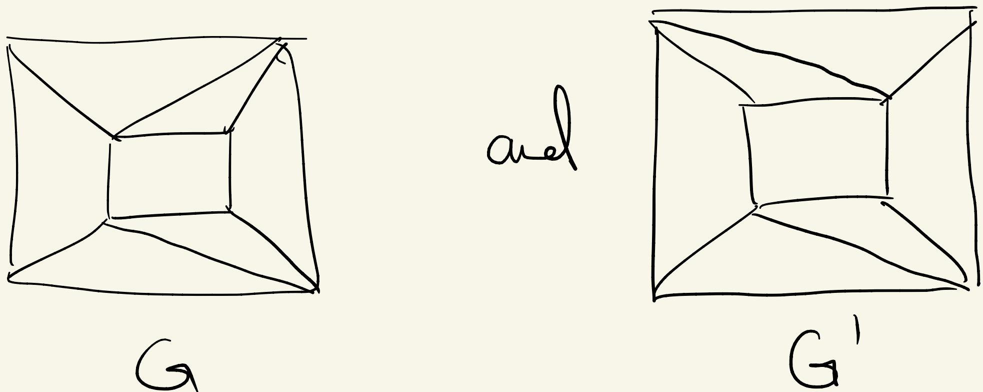
$\{i,j,k,l,m\} = \{1,\ldots,5\}$. and vertices



so that



<u>Def</u> Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there is a bijection $\varphi : V \to V'$ s.t. $\{u, v\} \in E \iff \{\varphi(u), \varphi(v)\} \in E'$.



$|E| = |E'|$ $\quad |V| = |V'|$ are $G \cong G'$?

Graph lingo:

If $\{u, v\} \in E$ we say u, v are neighboring or adjacent

If $u \in V$ and $k \in E$ is an edge $k = \{u, v\}$ we say u and k are incident (u is an end vertex of k)

The set of neighbors of u is denoted $N(u)$. The degree of u is $d(u) = |N(u)|$.

A vertex is isolated if $d(u) = 0$.

The degree sequence of G is $d_1 \geq d_2 \geq \dots \geq d_n$ where $d_i = d(u_i)$.

Isomorphic graphs have the same degree sequence.

Proposition (Thm 6.1). $G = (V, E)$ a graph then

$$\sum_{u \in V} d(u) = 2|E|$$

Proof $\displaystyle\sum_{u \in V} d(u) = \sum_{u} \Big(\sum_{\substack{K \in E \\ \text{s.t.} \\ K = \{u, v\}}} 1 \Big).$

$$= 2|E|$$

Corollary. every graph has an even # of vertices of odd degree.

Proof $\displaystyle\sum_{u \in V} d(u)$ is even by above.

$$\sum_{\substack{u \in V \\ d(u) \text{ even}}} d(u) + \sum_{\substack{u \in V \\ d(u) \text{ odd}}} d(u) \text{ is even}$$

For a sum of odd #s to be even there must be an even # of them. □.