30th April, 2020

MAT2250 Discrete Mathematics

Mandatory assignment 1 of 1

Submission deadline

April 2 nd of March 2020, 14:30 in Canvas (canvas.uio.no).

(**Changed from** Thursday 26 th of March 2020)

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

These solutions are mash up from those proposed in the assignments and those of my own. Special thanks to Jon Pål Hamre for sharing a latex file with solutions!

Problem 1. Let K_n be the complete graph on n vertices $\{v_1, \dots, v_n\}$.

1. How many spanning trees are there of K_6 which contain a single vertex of degree 5? Is there a spanning tree of K_6 with vertex degree sequence $(3, 3, 2, 1, 1, 1)$?

2. Let $d_1, \dots, d_n$ be a sequence of natural numbers each greater than or equal to 1 with $\sum_{i=1}^{n} d_i = 2n - 2$. Show that the number of spanning trees in K_n in which $\deg(v_i) = d_i$ for all i is equal to

$$\frac{(n-2)!}{(d_1 - 1)! \dots (d_n - 1)!}$$

   Hint: Adapt the bijection constructed in the lectures between spanning trees and sequences.

3. Consider the multivariable generating function where the sum is over all spanning trees T of K_n

$$\mathbf{T}(z_1, \dots, z_n) = \sum_{T} \prod_{i=1}^{n} z_i^{\deg(v_i)-1}.$$

   By the above exercise we have

$$\mathbf{T}(z_1, \dots, z_n) = \sum_{d_1, \dots, d_n} \frac{(n-2)!}{(d_1 - 1)! \dots (d_n - 1)!} z_1^{d_1 - 1} \dots z_n^{d_n - 1},$$

   where the sum is over all sequences of natural numbers $d_1, \dots, d_n$ which are greater than or equal to 1 satisfying $\sum_{i=1}^{n} d_i = 2n - 2$.

   Prove that $\mathbf{T}(z_1, \dots, z_n) = (z_1 + \dots + z_n)^{n-2}$ for all n.

Deduce the result proved in the lectures that the number of spanning trees of K_n is equal to n^{n-2}.

Solution to Problem 1.1

If a spanning tree of K_6 has a single vertex of degree 5 then all other vertices must be of degree 1 since $\sum_{i=1}^{6} \deg(v_i) = 2|E| = 2(|V| - 1) = 10$ and $\deg(v_i) > 0$ for all i. Therefore, once the vertex of degree 5 is chosen the tree is determined. There are 6 choices for the vertex of degree 6 and thus 6 such spanning trees.

Again a spanning tree of K_6 must satisfy $\sum_{i=1}^{6} \deg(v_i) = 2|E| = 2(|V|-1) = 10$. However if a spanning tree has degree sequence $(3,3,2,1,1,1)$ then the sum of these degrees is $12 \neq 10$ so there are no such spanning trees.

Solution to Problem 1.2

We adapt the proof enumerating the spanning trees of K_n from Aigner. The book outlines a bijection between spanning trees of K_n and sequences $\mathbf{a} = (a_1, \dots, a_{n-2})$ where $a_k \in \{1, \dots, n\}$. From the construction of the bijection, if a spanning tree of K_n has $\deg(v_i) = d_i$ then i appears exactly $d_i - 1$ times its sequence $\mathbf{a}$.

Therefore, the set of spanning trees of K_n with vertices of degrees $d_1, \dots, d_n$ are in bijection with sequences $(a_1, \dots, a_{n-2})$ where $a_k \in \{1, \dots, n\}$ and i appears exactly $d_i - 1$ times for all $i \in \{1, \dots, n\}$. The number of such sequences is equal to the number of orderings of the multi-set $\{1, \dots, 1, 2, \dots, 2, \dots, n, \dots, n\}$ where i appears d_i times. This is

$$\frac{(n-2)!}{(d_1-1)! \dots (d_n-1)!}.$$

Solution to Problem 1.3

We must show that

$$(z_1 + \dots + z_n)^{n-2} = \sum_{d_1, \dots, d_n} \frac{(n-2)!}{(d_1-1)! \dots (d_n-1)!} z_1^{d_1-1} \dots z_n^{d_n-1},$$

where the sum is over all sequences of natural numbers $d_1, \dots, d_n$ which are greater than or equal to 1 satisfying $\sum_{i=1}^{n} d_i = 2n-2$.

This follows from the so-called multinomial theorem: For any $k, n \in \mathbb{N}$ we have

$$(z_1 + \dots + z_n)^k = \sum_{\substack{i_1, \dots, i_n \\ i_1 + \dots + i_n = k}} \frac{k!}{i_1! \dots i_n!} z_1^{i_1} \dots z_n^{i_n}.$$

The above statement holds for all k when $n = 2$, this is the usual binomial theorem. Now suppose by induction that the statement holds for all k and for all $n' \leq n$. Then,

$$\begin{aligned}
(z_1 + \cdots + z_n)^k &= [z_1 + \cdots + z_{n-2} + (z_{n-1} + z_n)]^k \\
&= \sum_{\substack{i_1,\ldots,i_{n-2},j \\ i_1+\cdots+i_{n-2}+j=k}} \frac{k!}{i_1! \ldots i_{n-1}!} z_1^{i_1} \ldots z_{n-2}^{i_{n-2}} (z_{n-1} + z_n)^j \\
&= \sum_{\substack{i_1,\ldots,i_{n-2},j \\ i_1+\cdots+i_{n-1}=k}} \frac{k!}{i_1! \ldots i_{n-2}! j!} z_1^{i_1} \ldots z_{n-2}^{i_{n-2}} \sum_{\substack{i_{n-1},i_n \\ i_{n-1}+i_n=j}} \frac{j!}{i_n! i_{n-1}!} z_{n-1}^{i_{n-1}} z_n^{i_n} \\
&= \sum_{\substack{i_1,\ldots,i_n \\ i_1+\cdots+i_n=k}} \frac{k!}{i_1! \ldots i_n!} z_1^{i_1} \ldots z_n^{i_n}
\end{aligned}$$

This establishes the more general statement and we get

$$\mathbf{T}(z_1, \ldots, z_n) = (z_1 + \cdots + z_n)^{n-2} = \sum_{d_1,\ldots,d_n} \frac{(n-2)!}{(d_1-1)! \ldots (d_n-1)!} z_1^{d_1-1} \ldots z_n^{d_n-1}.$$

The number of spanning trees of K_n is simply the sum of all the coefficients of the generating series $\mathbf{T}(z_1, \ldots, z_n)$. This is obtained by substituting $z_i = 1$ for all i. So that $\mathbf{T}(1, \ldots, 1) = (1 + \cdots + 1)^{n-2} = n^{n-2}$.

An alternative proof of the multinomial theorem via multivariable generating functions: Consider the k fold product: $(z_1 + \cdots + z_n)(z_1 + \cdots + z_n) \ldots (z_1 + \cdots + z_n)$. We can think of the terms in the product as “bins" and the terms z_i as labelled balls contained in the bins. A term in the expansion of this product corresponds to picking a single labeled ball from each bin. When we choose z_j exactly i_j number of times this contributes to the monomial $z_1^{i_1} \ldots z_n^{i_n}$. Therefore, the coefficient of the monomial $z_1^{i_1} \ldots z_n^{i_n}$ corresponds to a choice of associating balls with prescribed labels to the k bins. In other words, it is the number of ways of orderings of the elements of the multiset $\{1, \ldots, 1, 2, \ldots, 2, \ldots, n, \ldots, n\}$ where j appears i_j times. This number is $\frac{k!}{i_1! \ldots i_n!}$.

Problem 2. Let $G = (V, E)$ be a graph with $|V| = n$. A map $f : V \to \{1, \ldots, t\}$ is called an admissible vertex labeling if $f(v) \neq f(v')$ whenever vv' is an edge of G. Let $P_G(t)$ denote the number of admissible vertex labelings of a graph G with t colours.

1. Calculate $P_G(t)$ for the path P_n with n vertices, the circuit C_n with n vertices, and the the complete graph K_n.

2. For a graph $G = (V, E)$ and an edge $k \in E$, let $G \backslash k = (V, E \backslash k)$ and let G/k denote the graph obtained from G by contracting the edge $k = uv$ and identifying the vertices u and v.[1] Prove that

$$P_G(t) = P_{G\backslash k}(t) - P_{G/k}(t).$$

Conclude from the above exercise that $P_G(t)$ is a polynomial in t for all G.

3. If G is the graph with n vertices and no edges then $P_G(t) = t^n$. Using this and the above formula, show that $P_G(t)$ has degree $n = |V|$ and the leading coefficient is 1.

Solution to Problem 2.1

1. Let $v_1, \ldots, v_n$ be the vertices of P_n so that v_i is adjacent to v_{i+1}. To obtain an admissible labelling of P_n we can begin by labeling v_1 by choosing any of the t labels. Then to label v_2 have only $t-1$ possibilities in order to remain admissible. Labeling the vertices in succession, we see that at each step there are exactly $t-1$ choices to label v_i for $i > 1$, since the vertex v_{i-1} has already taken on a label.

Therefore,

$$P_{P_n}(t) = t(t-1)^n.$$

For the circuit C_n, the same approach does not work so easily. We will prove that $P_{C_n}(t) = (t-1)^n + (-1)^n(t-1)$ by finding a recursive formula for the polynomial and then continuing by induction. Labelling the vertices in succession as above we arrive at a problem when we get to v_n. We need to know if v_1 and v_{n-1} have been given the same label or if they are different.

Case 1 : If v_1 and v_{n-1} have the same label then there are $t-1$ possible labels of v_n.

Case 2: If v_1 and v_{n-1} have different labels then there are $t-2$ possible labels of v_n

A labeling of v_1 through v_{n-1} satisfying case 1 would give us an admissible labeling of C_{n-2} by removing the vertex v_n and identifying v_{n-1} and v_1.

Therefore, we have the recurrence:

$$P_{C_n}(t) = (t-1)P_{C_{n-2}}(t) + (t-2)P_{C_{n-1}}(t).$$

Alternatively, we can use Problem 2.2 to derive the recurrence:

$$P_{C_n}(t) = P_{P_n}(t) - P_{C_{n-1}}(t). \tag{1}$$

[1]Notice that this operation might produce loops or multiple edges. If G has a loop then $P_G(t) = 0$ and it has multiple edges then $P_G(t) = P_{G'}(t)$ where G' is the graph with all but one of the multiple edges removed.

Using either of these recurrences we can prove by induction that $P_{C_n}(t) = (t-1)^n + (-1)^n(t-1)$. We will use Equation 1. The base case of the induction is when $n = 3$. Here we have

$$\begin{aligned} P_{C_3}(t) &= t(t-1)(t-2) \\ &= (t-1)[(t-1)^2 - 1] \\ &= (t-1)^2 - (t-1). \end{aligned}$$

So suppose the statement is true for all $k \leq n$. Then

$$\begin{aligned} P_{C_{n+1}}(t) &= P_{P_{n+1}}(t) - P_{C_n}(t) \\ &= t(t-1)^n - [(t-1)^n + (-1)^n(t-1)] \\ &= (t-1)^{n+1} + (-1)^{n+1}(t-1). \end{aligned}$$

For the complete graph K_n notice that starting from vertex v_1 we have t possible labels. Assigning labels to the vertices $v_2, \dots, v_n$ in succession we see that there are $t - i$ choices to assign a label to the vertex v_{i+1} since i labels have been used for the previous vertices moreover every pair of vertices are joined by an edge in the complete graph. Therefore we obtain

$$P_{K_n}(t) = t(t-1)\dots(t-n+1).$$

Solution to Problem 2.2

We will prove the equivalent statement that $P_{G\setminus k}(t) = P_G(t) + P_{G/k}(t)$. Consider the admissible labelings by the set $\{1, \dots, t\}$ of the graph $G \setminus k$. The set of admissible t-labelings $L_{G\setminus k}(t)$ admits a partition

$$L_{G\setminus k}(t) = S_{G\setminus k}(t) \sqcup D_{G\setminus k}(t),$$

where $S_{G\setminus k}(t)$ denotes the set of admissible labelings where u and v have the same label, and $D_{G\setminus k}(t)$ denotes the set of admissible labelings where u and v have different labels. By definition we have $P_{G\setminus k}(t) = |L_{G\setminus k}(t)|$. Moreover, labelings in $D_{G\setminus k}(t)$ correspond to admissible labelings of G, so $P_G(t) = |D_{G\setminus k}(t)|$. Finally, given a labeling in $S_{G\setminus k}(t)$ upon contracting the edge k we obtain an admissible labeling of G/k, so that $P_{G/k}(t) = |S_{G\setminus k}(t)|$. Combining all of this we obtain

$$P_{G\setminus k}(t) = |L_{G\setminus k}(t)| = |D_{G\setminus k}(t)| + |S_{G\setminus k}(t)| = P_G(t) + P_{G/k}(t),$$

which proves the claim.

To see that $P_G(t)$ is a polynomial we proceed by double induction on $n = |V|$ and $m = |E|$. For $m = 0$ and any n we have $P_G(t) = t^n$, which is a polynomial.

Assume the claim is proved for all n and all $m' < m$. If G has m edges, then $G\backslash k$ and G/k have $m-1$ edges so that both $PG\backslash k(t)$ and $P_{G/k}(t)$ are polynomials. The difference of two polynomials is a polynomial thus $P_G(t)$ is a polynomial.

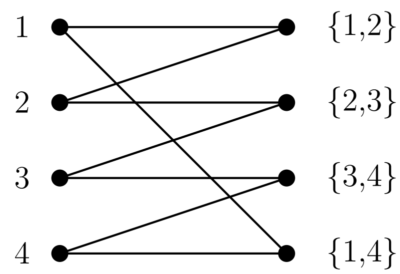
Solution to Problem 2.3

To see that $P_G(t)$ has degree $n = |V|$ and leading coefficient 1, we again proceed by double induction on n and $m = |E|$ If $m = 0$ the statement is true as above. Now we can assume the statement holds for all n and $m' < m$. Therefore, the polynomial $P_{G\backslash k}(t)$ has degree n and leading coefficient 1, whereas the polynomial $P_{G/k}(t)$ has degree $n-1$. Their difference $P_G(t)$ thus has degree n and leading coefficient 1.

Problem 3. 1. Let $T = \{1, 2, \ldots, 4\}$. Find the number of distinct transversals (or selection functions) of the family of sets

$$\mathcal{A} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}\}.$$

Following Aigner we consider the bipartite graph $G = (T + \mathcal{A}, E)$ where $E = \{tA_i \mid t \in A_i\}$. Therefore the graph G is



Following page 156 of Aigner *injective assignments* are injective maps of *subsets* of $\mathcal{A}$ to T. In this example there are exactly 2 injective assignments of all of $\mathcal{A}$ to T:

$$\{1,2\} \to 1, \{2,3\} \to 2, \{3,4\} \to 3, \{1,4\} \to 4$$

$$\{1,4\} \to 1, \{1,2\} \to 2, \{2,3\} \to 3, \{3,4\} \to 4.$$

The definition of *transversal* of page 156 is the same as the one on page 160: it is a subset of T which can be matched in the graph. Alternately, it is a subset $T_I = \{\phi(B) \in T \mid B \in \mathcal{B}$ where $\mathcal{B} \subset \mathcal{A}$ and $\phi : \mathcal{B} \to T$ is an injective assignment. Therefore there are 16 *transversals* and 2 *injective assignments*.

2. Suppose a bipartite graph $G = (S + T, E)$ is k-regular for $k \geq 1$ (i.e. $\deg(u) = k$ for all $u \in S \cup T$). Show that $|S| = |T|$ and G always contains a matching M with $|M| = |S| = |T|$.

   Solution to Problem 3.2 If G is a bipartite graph then $|E| = \sum_{u \in S} \deg(u) = \sum_{v \in T} \deg(v)$. Since G is k-regular we have $\deg(u) = \deg(v) = k$ for all $u \in S$ and $v \in T$. Therefore, $|E| = k|S| = k|T|$ and so $|S| = |T|$.

   To prove that G has a matching M with $|M| = |S| = |T|$ we use Hall's matching condition: G admits a matching with $|M| = |S|$ if and only if for all $A \subseteq S$ we have $|A| \leq |N(A)|$, where $N(A)$ is the set of neighbours of A.

   Let $A \subseteq S$, then there are $k|A|$ edges incident to the set A by the regularity of G. These $k|A|$ edges must be divided among the $|N(A)|$ vertices of $N(A)$. If $|N(A)| < |A|$ then there must be more than k edges incident to a vertex of $N(A)$ contradicting that G is k regular. Therefore, $|A| \leq |N(A)|$ for all $A \subseteq S$ and there exists a matching M of G with $|M| = |S|$.

3. Show that a k-regular bipartite graph $G = (S+T, E)$ contains at least $k!$ matchings with $|S| = |T|$ number of edges.

   Hint: Perform induction on $n = |S| = |T|$ and see Exercise 8.28 of Aigner.

   Solution to Problem 3.3 We will follow the hint from Aigner and establish the stronger statement for bipartite graphs with $\deg u \geq k$ for all $u \in S \cup T$. Let k be fixed. If $n < k$ G can not be k-regular, so the base step of the induction is when $n = k$.

   A k-regular bipartite graph with $|S| = |T| = n = k$ is the complete bipartite graph $K_{k,k}$, we know there is a one to one correspondence between the maximal matchings of $K_{k,k}$ and permutations of $\{1, 2, \ldots, k\}$. There are $k!$ such permutations so there are $k!$ maximal matchings of $K_{k,k}$.

   Suppose the statement is true for all k and all $n' < n + 1$ and we will establish the statement when $|S| = n + 1$. Let $G = (S \sqcup T, E)$ be a k-regular bipartite graph with $|S| = |T| = n + 1$. We want to show there exist at least $k!$ maximal matchings in G. Call a subset $A \subseteq S$ critical if $|A| = |N(A)|$. Both $\emptyset$ and S are critical.

   Case 1: Suppose there exist no proper non-empty subsets A of S which are critical. Then let $uv \in E$ be any edge of G and consider

$G' = G \setminus \{u, v\}$. G' is no longer k-regular, but every vertex have at least degree $k-1$ so by the induction hypothesis there is at least $(k-1)!$ matchings in G' of size n. We want to show that for any of these matchings there exists at least k matchings in G of size $n+1$.

Let M' be a matching of G' of size n. Then M' is not a maximal matching of G since u and v are not matched. By matching u and v we obtain a single maximal matching of G, namely $M = M' \cup k$. So each of the $(k-1)!$ matchings of G' can be extended to give a matching M of G with $|M| = n$. We want to show in fact that each such matching can be extended in k ways.

Label the vertices of S and T so that $u = u_{n+1}$ and $v = v_{n+1}$. Let the vertices adjacent to u in G be denoted $v_{j_1}, \dots, v_{j_k}$. For each $i_0 \in \{j_1, \dots, j_k\}$ our goal is to find an M'-alternating path

$$u, v_{i_0}, u_{i_0}, v_{i_1}, u_{i_1}, \dots, u_{i_l}, v$$

Recall that M' alternating means that $v_{I_k} u_{i_k} \in M'$ and $u_{i_k} v_{i_{k+1}} \in E \setminus M'$ for all $0 \leq k \leq l$. Then we can "flip" the alternating path as in Theorem 8.9 of Aigner to obtain a maximal matching M of G.

Two maximal matchings M and $\tilde{M}$ obtained in this way can only be equal if they came from the same choice of $i_0 \in \{j_1, \dots, j_k\}$, the same choice of alternating paths, and ultimately the same matching M' of G'. This shows that the $k(k-1)! = k!$ matchings of G are distinct and the theorem holds.

To find the M'-alternating path we use Algorithm 8.10 in Aigner starting to build the M' alternating tree from vertex u and choose vertex v_{i_0} for the vertex y the first time we perform step 2.

Case 2: If there exists some proper non-empty subset $A \subset S$ such that $|A| = |N(A)|$ then the graph G consists of two connected components, one of which is $G' = (A \sqcup N(A), E')$ of G consisting of only vertices from A and $N(A)$ and the edges E' between them. The graph G' is still k-regular and since A is a proper subset of S we have $|A| < n+1$ and we can apply the induction assumption to obtain $k!$ matchings of G' with $|A|$ edges.

The complement of G' in G is the graph $G'' = ((S \setminus A) + (T \setminus N(A), E'')$. This graph is also regular and $|S \setminus A| < n+1$ so there are again $k!$ matchings of G'' with $|S \setminus A|$ edges. Combining these we obtain at least $k!$ matchings of G.

This covers the two cases and completes the proof.