22$^{\text{nd}}$ March, 2020

MAT2250 Discrete Mathematics

Mandatory assignment 1 of 1

Submission deadline

April 2 $^{\text{nd}}$ of March 2020, 14:30 in Canvas (canvas.uio.no).

(**Changed from** Thursday 26 $^{\text{th}}$ of March 2020)

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let K_n be the complete graph on n vertices $\{v_1, \dots, v_n\}$.

1. How many spanning trees are there of K_6 which contain a single vertex of degree 5? Is there a spanning tree of K_6 with vertex degree sequence $(3, 3, 2, 1, 1, 1)$?

2. Let $d_1, \dots, d_n$ be a sequence of natural numbers each greater than or equal to 1 with $\sum_{i=1}^n d_i = 2n - 2$. Show that the number of spanning trees in K_n in which $\deg(v_i) = d_i$ for all i is equal to

$$\frac{(n-2)!}{(d_1 - 1)! \dots (d_n - 1)!}$$

   Hint: Adapt the bijection constructed in the lectures between spanning trees and sequences.

3. Consider the multivariable generating function where the sum is over all spanning trees T of K_n

$$\mathbf{T}(z_1, \dots, z_n) = \sum_T \prod_{i=1}^n z_i^{\deg(v_i)-1}.$$

   By the above exercise we have

$$\mathbf{T}(z_1, \dots, z_n) = \sum_{d_1, \dots, d_n} \frac{(n-2)!}{(d_1 - 1)! \dots (d_n - 1)!} z_1^{d_1 - 1} \dots z_n^{d_n - 1},$$

   where the sum is over all sequences of natural numbers $d_1, \dots, d_n$ which are greater than or equal to 1 satisfying $\sum_{i=1}^n d_i = 2n - 2$.

   Prove that $\mathbf{T}(z_1, \dots, z_n) = (z_1 + \dots + z_n)^{n-2}$ for all n.

Deduce the result proved in the lectures that the number of spanning trees of K_n is equal to n^{n-2}.

Problem 2. Let $G = (V, E)$ be a graph with $|V| = n$. A map $f : V \to \{1, \dots, t\}$ is called an admissible vertex labeling if $f(v) \neq f(v')$ whenever vv' is an edge of G. Let $P_G(t)$ denote the number of admissible vertex labelings of a graph G with t colours.

1. Calculate $P_G(t)$ for the path P_n with n vertices, the circuit C_n with n vertices, and the the complete graph K_n.

2. For a graph $G = (V, E)$ and an edge $k \in E$, let $G \backslash k = (V, E \backslash k)$ and let G/k denote the graph obtained from G by contracting the edge $k = uv$ and identifying the vertices u and v.[1] Prove that

$$P_G(t) = P_{G\backslash k}(t) - P_{G/k}(t).$$

Conclude from the above exercise that $P_G(t)$ is a polynomial in t for all G.

3. If G is the graph with n vertices and no edges then $P_G(t) = t^n$. Using this and the above formula, show that $P_G(t)$ has degree $n = |V|$ and the leading coefficient is 1.

Problem 3. 1. Let $T = \{1, 2, \dots, 4\}$. Find the number of distinct transversals (or selection functions) of the family of sets

$$\mathcal{A} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}.$$

2. Suppose a bipartite graph $G = (S + T, E)$ is k-regular for $k \geq 1$ (i.e. $\deg(u) = k$ for all $u \in S \cup T$). Show that $|S| = |T|$ and G always contains a matching M with $|M| = |S| = |T|$.

3. Show that a k-regular bipartite graph $G = (S + T, E)$ contains at least $k!$ matchings with $|S| = |T|$ number of edges.

   Hint: Perform induction on $n = |S| = |T|$ and see Exercise 8.28 of Aigner.

[1]Notice that this operation might produce loops or multiple edges. If G has a loop then $P_G(t) = 0$ and it has multiple edges then $P_G(t) = P_{G'}(t)$ where G' is the graph with all but one of the multiple edges removed.