MAT2250 Exercise list
Spring 2024 - Instructor: Kris Shaw

1. Week of January 15th - Fundamentals of counting and permutations

Warm ups.

(1) How many lists of length 4 without repetition can be formed from a set of size 6? What if repetitions are allowed?
(2) How many binary strings of length 5 are there containing at most three 0's?
(3) Draw the Young diagrams for all of the integer partitions of $n = 6$.
(4) How many ways are there of partitioning a set of size 6 into 3 subsets?
(5) Write the following permutations on 7 elements in the alternate forms (as words, bijective functions, and cycle notation)
  (a) The permutation of the word $\pi = 5673214$.
  (b) The permutation given by the map $\pi(i) = i + 1$ for $i < 7$ and $\pi(7) = 1$.
  (c) The permutation given in cycle notation $(1, 2)(4, 7)(5, 3)$.
(6) How many permutations on 6 elements are there of cycle type $1^3\ 3^1$?

Problem 1.1. Show that the binomial coefficients are "unimodal" by showing that they satisfy:

$$\binom{n}{0} < \binom{n}{1} < \binom{n}{2} \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil} > \cdots > \binom{n}{n}.$$

Problem 1.2. Let (n, m) be positive integers. A lattice path from $(0, 0)$ to (m, n) is a sequence of steps of length 1 either to the right or in the up direction (you aren't allowed to go down or to the left).

(1) How many distinct lattice paths are there from $(0, 0)$ to (m, n)? Hint: Write down the steps of a path in a word.
(2) Partition the set of lattice paths from $(0, 0)$ to (m, n) based on where they first meet the line $x = m$. Use this to come up with a summation formula for the number of paths found in Part (1).
(3) Show that the number of lattice paths from $(0, 0)$ to (m, n) which intersect the diagonal line from $(0, n)$ to $(n, 0)$ (i.e. the line given by the equation $x + y = n$) in the point $(k, n - k)$ is equal to $\binom{n}{k}\binom{m}{n-k}$. Use this to prove Vandermond's identity

$$\binom{m+n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{n-k}.$$

Problem 1.3. The Bell number $\tilde{B}_n$ is the number of all set partitions of a set of size n (i.e. into any number of non-empty subsets). Write down the first four Bell numbers by listing all of the set partitions. Show that

$$\tilde{B}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \tilde{B}_{n-k}.$$

Hint: Take the point of view of the set element $n+1$ and think about the size of the subset it ends up in.

Problem 1.4. Consider a permutation in word (one line) notation given by $\pi = a_1 a_2 a_3 \dots a_n$. Let b_j denote the number of elements to the left of j in the word π which are strictly bigger than j. Prove that a permutation π is determined by its sequence $b_1, \dots, b_n$.

Hints: As an example if $\pi = 1234\dots n$, then $b_j = 0$ for all j. Notice that $0 \leq b_j \leq n-j$ and that the number of sequences $b_1, \dots, b_n$ satisfying these inequalities is equal to $n!$.

Problem 1.5.

1) Construct a map from the set of permutations of $\{1, \dots, n\}$ with k cycles to the set of set partitions of $\{1, \dots, n\}$ with k parts.

2) Construct a map from the set of set partitions of $\{1, \dots, n\}$ with k parts to the set of integer partitions of n with k parts.

Hint: Remember to forget!

3) Show that the maps from (1) and (2) are surjective and conclude that $P_{n,k} \leq S_{n,k} \leq \sigma_{n,k}$ for all n, k.

4) Describe the composition of the two maps which sends the set of permutations of $\{1, \dots, n\}$ with k-cycles to integer partitions of n with k parts. What does this have to do with the cycle type of a partition.