UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in	MAT2400 — Real analysis
Day of examination:	Thursday, June 2, 2015
Examination hours:	14:30 – 18:30

This problem set consists of 3 pages.

Appendices:	None.
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let X be the space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the supremum metric

$$d_\infty(f,g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)| .$$

1a

Show that d_∞ defines a metric on X.

1b

Set $f_r(x) = f(x+r)$ for $r \in \mathbb{R}$. Show that if $f \in X$ and f is uniformly continuous, then $\lim_{r\to 0} d_\infty(f_r, f) = 0$.

1c

For $x \in \mathbb{R}$, let $g(x) = \cos(x^2\pi)$. Show that g is not uniformly continuous. (Hint: As x grows, g will oscillate more and more rapidly.)

1d

Is it true that $\lim_{r\to 0} d_\infty(f_r, f) = 0$ for all $f \in X$?

Problem 2

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function, and set

$$\operatorname{sign}(u) = \begin{cases} 1 & u > 0, \\ 0 & u = 0, \\ -1 & u < 0. \end{cases}$$

Show that the composite function $s(x) = \operatorname{sign}(f(x))$ is measurable.

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Problem 3

Let X be a vector space with norm $\|\cdot\|$, and let $V \subseteq X$ be a linear subspace (i.e., V is also a vector space with norm $\|\cdot\|$), such that $\text{int}(V) \neq \emptyset$. Show that $V = X$. ($\text{int}(V)$ is the set of interior points in V)

Problem 4

Let $C[0,1]$ denote the space of continuous functions from the interval $[0,1]$ with values in $\mathbb{R}$. Let Lu be defined by

$$(Lu)(t) = \int_0^1 \frac{1}{1+t+s} f(u(s))\, ds,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function.

4a

Show that L maps $C[0,1]$ into $C[0,1]$.

4b

Assume now that

$$|f(u) - f(v)| < \frac{1}{\ln(2)} |u - v| \quad \text{for all } u \text{ and } v.$$

Show that the equation $Lu = u$ has a unique solution in $C[0,1]$.

Problem 5

Let the function $f : [-\pi, \pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0, \\ 1 & x = 0, \end{cases}$$

and for x outside $[-\pi, \pi]$ f is the periodic extension.

5a

Show that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin(x)}{x}\, dx.$$

(Hint: write $\sin(x) = (e^{ix} - e^{-ix})/(2i)$ and use the change of variables $z = (n+1)x$ and $z = (n-1)x$.).

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5b

Use this to compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx.$$

THE END