Solutions to deferred exam in MAT2400, 2022

Problem 1. a) By definition of the directional derivative

$$F'(x;r) = \lim_{t\to 0} \frac{F(x+tr)-F(x)}{t}$$

$$= \lim_{t\to 0} \frac{\int_0^1 e^{-s}(x(s)+tr(s))^2\,ds - \int_0^1 e^{-s}x(s)^2\,ds}{t}$$

$$= \lim_{t\to 0} \frac{\int_0^1 e^{-s}\left(x(s)^2+2tx(s)r(s)+t^2r(s)^2\right)\,ds - \int_0^1 e^{-s}x(s)^2\,ds}{t}$$

$$= \lim_{t\to 0} \frac{\int_0^1 e^{-s}\left(2tx(s)r(s)+t^2r(s)^2\right)\,ds}{t}$$

$$= \lim_{t\to 0} \int_0^1 e^{-s}\left(2x(s)r(s)+tr(s)^2\right)\,ds = \int_0^1 e^{-s}2x(s)r(s)\,ds$$

b) We know that if F is differentiable, then $F'(x)(r) = F'(x;r) = \int_0^1 e^{-s}2x(s)r(s)\,ds$, and we only have to check that $F'(x;r)$ satisfies the conditions of a derivative.
If we write $A(r)$ for $F'(x;r)$, we first have to check that A is linear:

$$A(\alpha r+\beta u) = \int_0^1 e^{-s}2x(s)(\alpha r(s)+\beta u(s))\,ds$$

$$= \alpha\int_0^1 e^{-s}2x(s)r(s)\,ds + \beta\int_0^1 e^{-s}2x(s)u(s))\,ds = \alpha A(r)+\beta A(u).$$

Next we check that A is bounded:

$$|A(r)| = |\int_0^1 e^{-s}2x(s)r(s)\,ds| \le \int_0^1 e^{-s}2|x(s)||r(s)|\,ds \le \|r\|\int_0^1 e^{-s}2|x(s)|\,ds = K\|r\|,$$

where $K = \int_0^1 e^{-s}2|x(s)|\,ds$ is finite since x is bounded.

Finally, we must show that

$$\sigma(r) = F(x+r)-F(x)-A(r)$$

goes to 0 faster than r. We have

$$|\sigma(r)| = |\int_0^1 e^{-s}(x(s)+r(s))^2\,ds - \int_0^1 e^{-s}x(s)^2\,ds - \int_0^1 e^{-s}2x(s)r(s)\,ds|$$

$$= |\int_0^1 e^{-s}r(s)^2\,ds| \le \|r\|^2\int_0^1 e^{-s}\,ds \le M\|r\|^2.$$

where $M = \int_0^1 e^{-s}\,ds$. As this expression clearly goes to 0 faster than r, we have proved that F is differentiable with

$$F'(x)(r) = \int_0^1 e^{-s}2x(s)r(s)\,ds$$

Problem 2. a) Note that the series is geometric with first term $a_0 = 1$ and quotient $r = e^{-x}$. When $x > 0$, $e^{-x} < 1$, and the series converges. Hence

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1 - e^{-x}} = \frac{e^x}{e^x - 1}$$

For $x \in [a, \infty)$, we have $e^{-nx} \leq e^{-na}$, and hence Weierstrass's M-test with $M_n = e^{-an}$ shows that the series converges uniformly on $[a, \infty)$.

b) From a) we have

$$\int_a^b \sum_{n=0}^{\infty} e^{-nx}\, dx = \int_a^b \frac{e^x}{e^x - 1}\, dx \tag{1}$$

Since the series converges uniformly on $[a, b]$, it can be integrated termwise:

$$\int_a^b \sum_{n=0}^{\infty} e^{-nx}\, dx = \sum_{n=0}^{\infty} \int_a^b e^{-nx}\, dx = \int_a^b 1\, dx + \sum_{n=1}^{\infty} \int_a^b e^{-nx}\, dx$$

$$= b - a + \sum_{n=1}^{\infty} \left[-\frac{e^{-nx}}{n} \right]_a^b = b - a + \sum_{n=1}^{\infty} \frac{e^{-na} - e^{-nb}}{n}$$

On the other hand, the right hand side of (1) can be integrated by the substitution $u = e^x$:

$$\int_a^b \frac{e^x}{e^x - 1}\, dx = \int_{e^a}^{e^b} \frac{1}{u+1}\, du = \Big[\ln(u+1) \Big]_{e^a}^{e^b} = \ln(e^b + 1) - \ln(e^a + 1).$$

Combining what we now have, we get

$$b - a + \sum_{n=1}^{\infty} \frac{e^{-na} - e^{-nb}}{n} = \ln(e^b + 1) - \ln(e^a + 1),$$

which is equivalent to

$$\sum_{n=1}^{\infty} \frac{e^{-na} - e^{-nb}}{n} = \ln(e^b - 1) - \ln(e^a - 1) + a - b.$$

Problem 3. a) We need to show that the three axioms for norms are satisfied:

(i) $\|f\| \geq 0$ with equality if and only if $f = 0$.

(ii) $\|\alpha f\| = |\alpha| \|f\|$.

(iii) $\|f + g\| \leq \|f\| + \|g\|$.

We have:

(i) By definition, $\|f\| \geq 0$ and $\|0\| = 0$. If $f \neq 0$, there is an a such that $f(a) \neq 0$, and hence

$$\|f\| = \sum_{m=0}^{\infty} |f(m)| \geq |f(a)| > 0.$$

(ii) We have

$$\|\alpha f\| = \sum_{m=0}^{\infty} |\alpha f(m)| = |\alpha| \sum_{m=0}^{\infty} |f(m)| = |\alpha| \|f\|.$$

(iii) We have

$$\|f + g\| = \sum_{m=0}^{\infty} |f(m) + g(m)| \leq \sum_{m=0}^{\infty} (|f(m)| + |g(m)|)$$

$$= \sum_{m=0}^{\infty} |f(m)| + \sum_{m=0}^{\infty} |g(m)| = \|f\| + \|g\|.$$

b) Note that if $n > k$, then (summing a geometric series)

$$\|f_n - f_k\| = \sum_{m=0}^{\infty} |f_n(m) - f_k(m)| = \sum_{k+1}^{n} 2^{-m} < \sum_{k+1}^{\infty} 2^{-m} = \frac{2^{-(k+1)}}{1 - \frac{1}{2}} = 2^{-k}$$

As we can clearly get 2^{-k} as small as we want by choosing k large enough, $\{f_n\}$ is a Cauchy sequence.

To show that X isn't complete, it suffices to show that the Cauchy sequence $\{f_n\}$ doesn't converge. Assume for contradiction that $\{f_n\}$ converges to an element $f \in X$. Since f only has finitely many nonzero values, there is a largest number k such that $f(k) \neq 0$. This means that for any $n > k$, we have

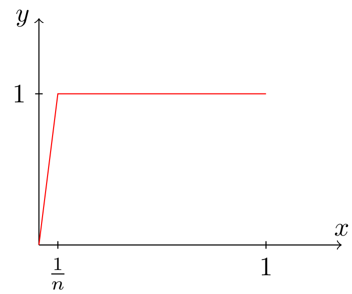
$$\|f_n - f\| = \sum_{m=0}^{\infty} |f_n(m) - f(m)| \geq |f_n(k+1) - f(k+1)| = 2^{-(k+1)},$$

and hence $\{f_n\}$ cannot converge to f.

Problem 4. a) The function $f\colon K \to \mathbb{R}$ defined by $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{a}\|$ is continuous. Since K is compact by the Bolzano-Weierstrass Theorem, f has a minimum point $\mathbf{x}$ by the the Extreme Value Theorem.

b) Since K is nonempty, there must be an element $\mathbf{b}$ in K. Let $r = \|\mathbf{b} - \mathbf{a}\|$ and let $\overline{B}(\mathbf{a}; r)$ be the *closed* ball of radius r around $\mathbf{a}$. The set $K \cap \overline{B}(\mathbf{a}; r)$ is nonempty as it contains $\mathbf{b}$, and it is compact by the Bolzano-Weierstrass Theorem. By part a) there is an element $\mathbf{x}$ in $K \cap \overline{B}(\mathbf{a}; r)$ that is nearest to $\mathbf{a}$. As all points in $K \setminus \overline{B}(\mathbf{a}, r)$ are more than a distance r away from $\mathbf{a}$, this point must also be the nearest point to $\mathbf{a}$ in K.

Problem 5. a) The figure shows the graph of f_n.



As $\|f_n\| = 1$ for all n, the set is obviously bounded. To prove that it is closed, it suffices to show that if g isn't one of the f_n's, then there is an $\epsilon > 0$ such that $B(g, \epsilon)$ doesn't contain any f_n. First note that if g is constant 1, then $\|g - f_n\| = 1$ for all n, and hence we can take $\epsilon = 1$. If g is not constant 1, there is an $a > 0$ such that $g(a) \neq 1$. This means that if n is so large than $\frac{1}{n} < a$, then $\|g - f_n\| \geq |g(a) - 1|$. As there are only finitely many n's that do not satisfy $\frac{1}{n} < a$, there is an $\epsilon > 0$ such that $B(g, \epsilon)$ doesn't contain any of these. If we also make sure that $\epsilon < |g(a)-1|$, we have obtained what we want.

b) By Arzela-Ascoli's Theorem, A is compact if and only if it is bounded, closed and equicontinuous. As we have checked that A is closed and bounded, equicontinuity is the crucial property. To see that the sequence $\{f_n\}$ isn't equicontinuous, choose $\epsilon = \frac{1}{2}$ and note that no matter how small $\delta > 0$ is, there will be an n such that $\frac{1}{n} < \delta$, and then $|f_n(\frac{1}{n}) - f_n(0)| = 1 > \epsilon$ even though $|\frac{1}{n} - 0| < \delta$. Hence there is no δ that works for all n, and hence $\{f_n\}$ isn't equicontinuous. By Arzela-Ascoli's Theorem, A isn't compact.