MAT2400: Solutions Spring 2010

Problem 1

a) Since d_1, d_2 are metrics, we get that

$$d(x,y) = 0 \Leftrightarrow d_1(x,y) = d_2(x,y) = 0 \Leftrightarrow x = y,$$

Moreover it also follows that $d(x,y) = d(y,x)$. Let $x, y, z \in X$. We may assume that $d(x,y) = d_1(x,y)$. Then

$$d(x,y) = d_1(x,y) \leq d_1(x,z) + d_1(z,y) \leq d(x,z) + d(z,y),$$

All together, this proves that d is a metric on X.

b) If f is not uniformly continuous, there exists $\epsilon > 0$ such that for any $\delta > 0$, we may find $x = x(\delta), y = y(\delta)$ such that $d(x,y) < \delta$, but $|f(x) - f(y)| \geq \epsilon$. Putting $\delta = \frac{1}{n}$, $x_n = x(\frac{1}{n}), y_n = y(\frac{1}{n})$, $n = 1, 2 \ldots$, we get $d(x_n, y_n) < \frac{1}{n}$, hence $d(x_n, y_n) \to 0$ as $n \to \infty$, but $|f(x_n) - f(y_n)| \geq \epsilon$.

c) Let us suppose that f is continuous but not uniformly continuous. By b) above, we may find $\epsilon > 0$ and sequences x_n, y_n such that $d(x_n, y_n) \to 0$ as $n \to \infty$, but $|f(x_n) - f(y_n)| \geq \epsilon$ for each n. Since X satisfies the Bolzano-Weierstrass condition, we may find a subsequence $x_{n(j)}$ of x_n which converges to a point $x \in X$. Let $y_{n(j)}$ be the induced subsequence of y_n. Then $d(x_{n(j)}, y_{n(j)}) \to 0$ as $j \to \infty$. Since, $d(x, y_{n(j)}) \leq d(x, x_{n(j)}) + d(x_{n(j)}, y_{n(j)})$, and $d(x, x_{n(j)}) \to 0$ as $j \to \infty$, we must also have that $y_{n(j)} \to x$ as $j \to \infty$. Since f is continuous, we thus get that $|f(x_{n(j)}) - f(x)|, |f(y_{n(j)}) - f(x)| \to 0$ as $j \to \infty$. Since

$$|f(x_{n(j)}) - f(y_{n(j))})| \leq |f(x_{n(j)}) - f(x)| + |f(x) - f(y_{n(j)})|,$$

we get that $|f(x_{n(j)}) - f(y_{n(j)}| \to 0$ as $j \to \infty$. Now, our assumption that $|f(x_n) - f(y_n)| \geq \epsilon$ for each n, implies that $|f(x_{n(j)}) - f(y_{n(j)})| \geq \epsilon$ for each j, contradicting that $|f(x_{n(j)}) - f(y_{n(j)})| \to 0$. So f must be uniformly continuous.

Problem 2

a) Assume that $M = \sup A$. Then M is an upper bound for A, hence $a \leq M$ for each $a \in A$. Since A is the least upper bound for A, $M - \frac{1}{n}$ is not an upper bound for A for any $n = 1, 2, \ldots$. So for each n, we can find a_n in A such that $M - \frac{1}{n} < a_n \leq M$. This gives us a sequence $a_n \in A$ such that $a_n \to M$ as $n \to \infty$.

Assume that $a \leq M$ for each $a \in A$ and that we have a sequence $a_n \in A$ such that $a_n \to M$ as $n \to \infty$. Since $a \leq M$ for each $a \in A$, M is an upper bound for A. Let $N < M$. Then, if n is big enough, we must have that $N < a_n \leq M$, showing that N cannot be an upper bound for A. It follows that M is the least upper bound, i.e. $M = \sup A$.

b) We know that $t \to e^{-t}$ is decreasing in $[0, \infty)$ and $e^{-t} \to 0$ as $t \to \infty$. Let $\epsilon > 0$. Then we can find a positive integer N such that $e^{-N^2} < \epsilon$. Let $n \geq N$ and $|x| \geq n$. Then $0 \leq f(x) = e^{-x^2} \leq e^{-n^2} \leq e^{-N^2} < \epsilon$, and $0 \leq f_n(x) \leq f_n(\pm n) = e^{-n^2} \leq e^{-N^2} < \epsilon$. This implies that $|f(x) - f_n(x)| < \epsilon$ when $n \geq N$ and $|x| \geq n$. When

$|x| < n$, we get that $|f(x) - f_n(x)| = 0$. So for any x, and $n \geq N$, we get that $|f(x) - f_n(x)| < \epsilon$, proving that f_n converges uniformly to f.

c) Let $f \in C_K(\mathbb{R})$, and consider $a = a(f), b = b(f)$. We know that a continuous function defined on a closed bounded interval is bounded. Therefore, the restriction of f to $[a, b]$ is bounded. Now $f \equiv 0$ outside $[a, b]$, and f must therefore be bounded on $\mathbb{R}$. That is, $f \in B_C(\mathbb{R})$.

We know that convergence in the uniform metric is the same as uniform convergence. Now, the sequence in b) above is a sequence of functions in $C_K(\mathbb{R})$ which is converging uniformly (hence converging in the uniform metric) to a function in $B_C(\mathbb{R}) - C_K(\mathbb{R})$. This proves that $C_K(\mathbb{R})$ is not a closed subspace of $B_C(\mathbb{R})$.

d) If f vanishes outside an interval $[a, b]$, gf must vanish outside $[a, b]$. Also gf is continuous if f and g are continuous. So $T_g(f) \in C_K(\mathbb{R})$ if $f \in C_K(\mathbb{R})$. It is also obvious that $T_g(f_1 + f_2) = g(f_1 + f_2) = gf_1 + gf_2 = T_g(f_1) + T_g(f_2)$, and that $T_g(\lambda f) = \lambda gf = \lambda T_g(f)$. so

$$T_g : C_K(\mathbb{R}) \to C_K(\mathbb{R})$$

is a linear map. Moreover, we have that

$$|g(x)f(x)| \leq \sup\{|g(y)| \,:\, y \in \mathbb{R}\} \sup\{|f(y)| \,:\, y \in \mathbb{R}\} = ||g||_\infty ||f||_\infty.$$

This proves that $||T_g(f)||_\infty \leq ||g||_\infty ||f||_\infty$, and from this inequality follows that T_g is continuous. Also, since $||T_g|| = \sup\limits_{||f||_\infty \leq 1} ||T_g(f)||_\infty$, and $||T_g(f)||_\infty \leq ||g||_\infty ||f||_\infty \leq ||g||_\infty$, when $||f||_\infty \leq 1$, we get that $||T_g|| \leq ||g||_\infty$.

To prove that, $||T_g|| \geq ||g||_\infty$, let $\epsilon > 0$. We may find $x \in \mathbb{R}$ such that $||g||_\infty \geq |g(x)| > ||g||_\infty - \epsilon$ (this follows from a) above). Let $[a, b]$ be a closed interval such that $x \in (a, b)$. Let f be a continuous function such that $f(t) = 0$ outside $[a, b]$, $0 \leq f(t) \leq 1$ for all t and $f(x) = 1$ (we may choose such f beeing linear on $[a, x]$ and $[x, b]$). Then $||f||_\infty = f(x) = 1$, and $||T_g(f)||_\infty \geq |gf(x)| = |g(x)| \geq ||g||_\infty - \epsilon$. From the definition of $||T_g||$ we get that $||T_g|| \geq ||T_g(f)||_\infty \geq ||g||_\infty - \epsilon$. It follows that $||T_g|| \geq ||g||_\infty$, hence $||T_g|| = ||g||_\infty$.

Problem 3

a) Let $x = r\cos\theta$, $y = r\sin\theta$. When $(x, y) \neq (0, 0)$, we get that

$$|f(x, y)| = |\frac{r^3 \cos\theta \sin^2\theta}{r^2}| \leq r,$$

and we see that $|f(x, y)| \to 0$ as $r \to 0$. This shows that f is continuous at $(0, 0)$.

We have $f(x, 0) = f(0, y) = 0$, and this gives immediately that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0, 0)$ and are both equal 0. From this follows that if f is differentiable at $(0, 0)$, the derivative of f at this point must be the zero-mapping. Assume that f is differentiable at $(0, 0)$. Then, writing $f(x, y) = \epsilon(x, y)||(x, y)||$, we get that $\epsilon(x, y) = \frac{xy^2}{(x^2+y^2)^{\frac{3}{2}}}$ and we must have that $\epsilon(x, y) \to 0$ as $||(x, y)|| \to 0$. We see however that $\epsilon(t, t) = \frac{1}{2\sqrt{2}} \nrightarrow 0$ as $t \to 0$. This shows that f is not differentiable at $(0, 0)$.

b) The Jacobian matrix of f is given by

$$DF(x,y,z) = \begin{pmatrix} e^{x-y} & -e^{x-y} & 2z \\ e^{x+y} & e^{x+y} & \frac{2z}{1+z^2} \\ 0 & 0 & 1 \end{pmatrix}.$$

F is obviously continuously differentiable at $\mathbb{R}^3$. We see that $\det DF(x,y,z) = 2e^{2x} \neq 0$. Especially we get that $DF(0,0,0)$ is invertible. It follows from the inverse function theorem that there exist open sets B and V, with $(0,0,0) \in B$ and $F(0,0,0) = (1,1,0) \in V$ such that $F|B : B \to V$ is a differentiable bijection with differentiable inverse $G : V \to B$. Writing $G(u,v,w) = (f(u,v,w), g(u,v,w), h(u,v,w))$, we get that $(u,v,w) = F(G(u,v,w)) = (\ldots, \ldots, h(u,v,w))$ so $h(u,v,w) = w$.

We see from above that

$$DF(0,0,0) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By inspection, we see that

$$DG(1,1,0) = DF(0,0,0)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$